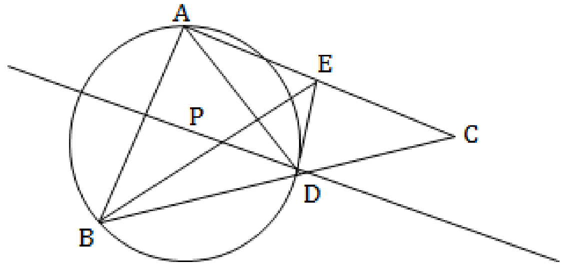
RMO 2016

1. Let ABC be a triangle and D be the mid-point of BC. Suppose the angle bisector of $\angle ADC$ is tangent to the circumcircle of triangle ABD at D. Prove that $\angle A = 90^0$.

Solution:

Let P bc the center of the circumcircle Γ of ΔABC. Let the tangent at D to Γ intersect AC in E. Then PD $\perp$ DE. Since DE bisects $\angle$ ADC, this implies that DP bisects $\angle$ ADB. Hence the circumcenter and the incenter of ΔABD lies on the same line DP. This implies that DA = DB. Thus DA = DB = DC and hence D is the circumcenter of ΔABC. This gives $\angle A = 90^0$.



2. Let a, b, c be positive real numbers such that abc = 1. Prove that

$$\frac{a^3}{(a-b)(a-c)} + \frac{b^3}{(b-c)(b-a)} + \frac{c^3}{(c-a)(c-b)} \geq 3.$$

Solution:

Observe that

$$\frac{1}{(a-b)(a-c)} = \frac{(b-c)}{(a-b)(b-c)(a-c)}$$

$$= \frac{(a-c)-(a-b)}{(a-b)(b-c)(a-c)} = \frac{1}{(a-b)(b-c)} - \frac{1}{(b-c)(a-c)}.$$

Hence

$$\frac{a^3}{(a-b)(a-c)} + \frac{b^3}{(b-c)(b-a)} + \frac{c^3}{(c-a)(c-b)} = \frac{a^3-b^3}{(a-b)(b-c)} + \frac{c^3-a^3}{(c-a)(c-b)}$$

$$= \frac{a^2+ab+b^2}{b-c} - \frac{c^2+ca+a^2}{b-c}$$

$$= \frac{ab+b^2-c^2-ca}{b-c}$$

$$= \frac{(a+b+c)(b-c)}{b-c} = a+b+c.$$

Therefore

$$\frac{a^3}{(a-b)(a-c)} + \frac{b^3}{(b-c)(b-a)} + \frac{c^3}{(c-a)(c-b)} = a+b+c \geq 3(abc)^{1/3} = 3.$$

3. Let a, b, c, d, e, f be positive integers such that $\frac{a}{b} < \frac{c}{d} < \frac{e}{f}$.

Suppose a f – be = – 1. Show that $d \geq b+f$.

Solution:

Since bc – ad > 0, we have bc – ad ≥ 1. Similarly, we obtain de – fc ≥ 1. Therefore
d = d (be – af) = dbe – daf = dbe – bfc + bfc – adf = b (de – fc) + f(bc – ad) ≥ b + f.

4. There are 100 countries participating in an Olympiad. Suppose n is a positive integer such that each of the 100 countries is willing to communicate in exactly n languages. In each set of 20 countries can communicate in at least one common language, and no language is common to all 100 countries, what is the minimum possible value of n?

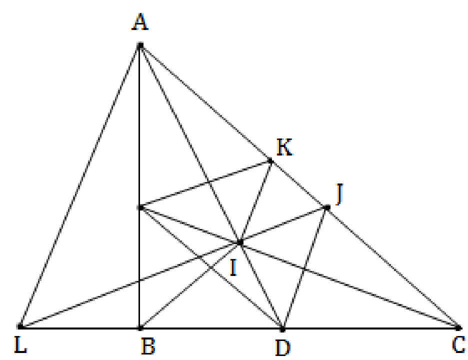
Solution:

We show that n = 20. We first show that n = 20 is possible. Call the countries C_1, ..., C_{100}. Let 1, 2, ..., 21 be languages and suppose, the country $C_i\left(1 \le i \le 20\right)$ communicates exactly in the languages $\{j : 1 \le j \le 20, j \ne i\}$. Suppose each of the countries C_{21}, ..., C_{100} communicates in the languages 1, 2, ..., 20. Then, clearly every set of 20 countries have a common language of communication.
Now, we show that n ≥ 20. If n < 20, look at any country A communicating in the languages $L_1,\ldots,L_n$. As no language is common to all 100 countries, for each Li, there is a country Ai not communicating in Li. Then, the $n+1 \le 20$ countries A, A_1, A_2, _ _ _ , A_n have no common language of communication. This contradiction shows $n \ge 20$.

5. Let ABC be a right-angled triangle with $\angle B = 90^0$. Let I be the incentre of ABC. Extend AI and CI; let them intersect BC in D and AB in E respectively. Draw a line perpendicular to AI at I to meet AC in J; draw a line perpendicular to CI a I to meet AC in K. Suppose DJ = EK. Prove that BA = BC.

Solution:

Extend JI to meet CB extended at L. Then $\angle BLI = \angle BAI = \angle IAC$. Therefore $\angle LAD = \angle IBD = 45^0$. Since $\angle AIL = 90^0$, it follows that $\angle ALI = 45^0$. Since $\angle AIL = 90^0$, it follows that $\angle ALI = 45^0$. Therefore AI = IL. This shows that the triangles AIJ and LID are congruent giving IJ = ID. Similarly, IK = IE. Since IJ ⊥ ID and IK ⊥ IE and since DJ = EK, we see that IJ = ID = IK = IE. Thus E, D, J, K are concyclic. This implies together with DJ = EK that EDJK is an isosceles trapezium. Therefore DE□JK. Hence $\angle EDA = \angle DAC = \angle A/2$ and $\angle DEC = \angle ECA = \angle C/2$. Since IE = ID, it follows that $\angle A/2 = \angle C/2$. This means BC = BC.



6. (a) Given any natural number N ≥ 3, prove that there exists a strictly increasing sequence of N positive integers in harmonic progression.

(b) Prove that there cannot exist a strictly increasing infinite sequence of positive integers which is in harmonic progression.

Solution:

(a) Let N ≥ 3 be a given positive integer. Consider the HP

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{N}.$$

If we multiply this by N!, we get the HP

$$N!, \frac{N!}{2}, \frac{N!}{3}, \frac{N!}{4}, \ldots, \frac{N!}{N}.$$

This is decreasing. We write this in reverse order to get a strictly increasing HP:

$$\frac{N!}{N}, \frac{N!}{N-1}, \frac{N!}{N-2}, \ldots, \frac{N!}{3}, \frac{N!}{2}, N!.$$

(b) Assume the contrary that there is an infinite strictly increasing sequence (a_1, a_2, a_3, ...,) of positive integers which forms a harmonic progression. Define $b_k = 1/a_k$, for $k \ge 1$. Then (b_1, b_2, b_3, ...) is a strictly decreasing sequence of positive rational numbers which is in an arithmetic progression.

Let $d = b_2 - b_1 < 0$ be its common difference. Then $b_1 - b_2 > 0$. Choose a positive integer n such that

$$n > \frac{b_1}{b_1 - b_2}.$$

Then

$$b_{n+1} = b_1 + (b_2 - b_1)n = b_1 - (b_1 - b_2)n < b_1 - \left(\frac{b_1}{b_1 - b_2}\right) \times (b_1 - b_2) = 0.$$

Thus for all large n, we see that b_n is negative contradicting b_n is positive for all n.