JEE (Mains) Mathematics Solution

25.2.2021 (Shift-2)

Section-I

Multiple Choice Questions: This section contains 20 multiple choice questions. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.

1. If $I_n = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^n x dx$, then :

(a) $\frac{1}{I_2+I_4}, \frac{1}{I_3+I_5}, \frac{1}{I_4+I_6}$ are in G.P. (b) $I_2 + I_4, I_3 + I_5, I_4 + I_6$ are in A.P.

(c) $I_2 + I_4, (I_3 + I_5)^2, I_4 + I_6$ are in G.P. (d) $\frac{1}{I_2+I_4}, \frac{1}{I_3+I_5}, \frac{1}{I_4+I_6}$ are in A.P.

Ans. (d)

Solution:

$$I_n = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^n x dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^{n-2} x\left(\cot^2 x\right)dx$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^{n-2} x\left(\operatorname{cosec}^2 x - 1\right)dx$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^{n-2} x \operatorname{cosec}^2 x \, dx - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^{n-2} x dx$$

$$= -\left.\frac{\cot^{n-1} x}{n-1}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} - I_{n-2} = \frac{1}{n-1} - I_{n-2}$$

$$I_n + I_{n-2} = \frac{1}{n-1} \Rightarrow \frac{1}{I_{n-2}+I_n} = n-1 \text{ = a linear expression in n.}$$

$\therefore$ Sequence $\frac{1}{I_{n-2}+I_n}$ is an A.P.

2. Let α and β be the roots of $x^2 - 6x - 2 = 0$. If $a_n = \alpha^n - \beta^n$ for $n \geq 1$,

then the value of $\frac{a_{10} - 2a_8}{3a_9}$ is :

(a) 2 (b) 4 (c) 3 (d) 1

Ans. (a)

Solution:

α, β are roots of $x^2 - 6x - 2 = 0$

$\therefore \alpha^2 - 6\alpha - 2 = 0$

$\Rightarrow \alpha^2 - 2 = 6\alpha$

Similarly $\beta^2 - 2 = 6\beta$

$$\frac{a_{10}-2a_8}{3a_9}=\frac{\alpha^{10}-\beta^{10}-2\left(\alpha^8-\beta^8\right)}{3\left(\alpha^9-\beta^9\right)}$$

$$=\frac{\left(\alpha^{10}-2\alpha^8\right)-\left(\beta^{10}-2\beta^8\right)}{3\left(\alpha^9-\beta^9\right)}$$

$$=\frac{\alpha^8\left(\alpha^2-2\right)-\beta^8\left(\beta^2-2\right)}{3\left(\alpha^9-\beta^9\right)}=\frac{\alpha^8(6\alpha)-\beta^8(6\beta)}{3\left(\alpha^9-\beta^9\right)}$$

$$=\frac{6\left(\alpha^9-\beta^9\right)}{3\left(\alpha^9-\beta^9\right)}=2$$

3. A hyperbola passes through the foci of the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$ and its transverse and conjugate axes coincide with major and minor axes of the ellipse, respectively. If the product of their eccentricities is one, then the equation of the hyperbola is:

(a) $\frac{x^2}{9}-\frac{y^2}{16}=1$ (b) $x^2 - y^2 = 9$ (c) $\frac{x^2}{9}-\frac{y^2}{4}=1$ (d) $\frac{x^2}{9}-\frac{y^2}{25}=1$

Ans. (a)

Solution:

Eccentricity of Ellipse $e_1 = \sqrt{1-\frac{16}{25}}=\frac{3}{5}$

Foci = $(\pm ae, 0)=(\pm 3, 0)$

For Hyperbola

Eccentricity $e_2 = \frac{5}{3}$

Semi-transverse axis $\rightarrow a=3$

$$b^2=a^2\left(e^2-1\right)=9\left(\frac{25}{9}-1\right)=16$$

Equation of Hyperbola

$$\frac{x^2}{9}-\frac{y^2}{16}=1$$

4. Let x denote the total number of one-one functions from a set A with 3 elements to a set B with 5 elements and y denote the total number of one-one functions from the set A to the set $A \times B$. Then:

(a) 2y = 273x (b) 2y = 91x (c) y = 273x (d) y = 91x

Ans. (b)

Solution:

n(A) = 3, n(B) = 5

$$x={}^5C_3\times 3!=5\times 4\times 3$$

$$n(A\times B)=15$$

$$y={}^{15}C_3\times 3!=15\times 14\times 13$$

$$\frac{y}{x}=\frac{15\times 14\times 13}{5\times 4\times 3}=\frac{91}{2}$$

2y = 91x

5. A function f(x) is given by $f(x) = \dfrac{5^x}{5^x + 5}$, then the sum of the series

$f\left(\frac{1}{20}\right)+f\left(\frac{2}{20}\right)+f\left(\frac{3}{20}\right)+.....+f\left(\frac{39}{20}\right)$ is equal to:

(a) $\frac{19}{2}$ (b) $\frac{29}{2}$ (c) $\frac{49}{2}$ (d) $\frac{39}{2}$

Ans. (d)

Solution:

$$f(2-x)=\frac{5^{2-x}}{5^{2-x}+5}=\frac{5^x}{5+5^x}$$

So f(x) + f(2 – x) = 1

$$\sum_{r=1}^{39} f\left(\frac{r}{20}\right)=\sum_{r=1}^{19}\left(f\left(\frac{r}{20}\right)+f\left(2-\frac{r}{20}\right)\right)+f(1)$$

$$= 19+\frac{1}{2}=\frac{39}{2}$$

6. The minimum value of $f(x) = a^{a^x} + a^{1-a^x}$, where $a, x \in R$ and a > 0, is equal to:

(a) a + 1 (b) $2\sqrt{a}$ (c) $a+\frac{1}{a}$ (d) 2a

Ans. (b)

Solution:

$$f(x)=a^{a^x}+\frac{a}{a^{a^x}}$$

$$\because \frac{a^{a^x}+\frac{a}{a^{a^x}}}{2} \ge \sqrt{a^{a^x}.\frac{a}{a^{a^x}}}$$

$$\Rightarrow f(x)\ge 2\sqrt{a}$$

$$f(x)_{min} = 2\sqrt{a}$$

7. If the curve $x^2 + 2y^2 = 2$ intersects the line x + y = 1 at two points P and Q, then the angle subtended by the line segment PQ at the origin is:

(a) $\frac{\pi}{2}-\tan^{-1}\left(\frac{1}{4}\right)$ (b) $\frac{\pi}{2}-\tan^{-1}\left(\frac{1}{3}\right)$ (c) $\frac{\pi}{2}+\tan^{-1}\left(\frac{1}{3}\right)$ (d) $\frac{\pi}{2}+\tan^{-1}\left(\frac{1}{4}\right)$

Ans. (d)

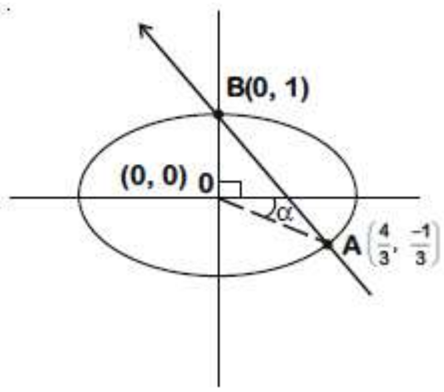
Solution:

$y = 1 - x$ ...(i)

$x^2 + 2y^2 = 2$ ...(ii)

$\Rightarrow x^2 + 2(1 - x)^2 = 2$

$3x^2 - 4x = 0$



$$x=0,\frac{4}{3}$$

$$y=1,\frac{-1}{3}$$

$$B(0, 1),\ A\left(\frac{4}{3},\frac{-1}{3}\right)$$

$$\tan\alpha = \frac{\frac{1}{3}}{\frac{4}{3}} = \frac{1}{4} \Rightarrow \alpha = \tan^{-1}\frac{1}{4}$$

$$\angle AOB = \frac{\pi}{2} + \tan^{-1}\frac{1}{4}$$

8. $\lim_{n\to\infty}\left[\frac{1}{n} + \frac{n}{(n+1)^2} + \frac{n}{(n+2)^2} + ... + \frac{n}{(2n-1)^2}\right]$ is equal to:

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) 1 (d) $\frac{1}{4}$

Ans. (a)

Solution:

$$\lim_{n\to\infty}\sum_{r=0}^{n-1}\frac{n}{(n+r)^2}$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n-1}\frac{1}{\left(1+\frac{r}{n}\right)^2}$$

$$\int_0^1 \frac{dx}{(1+x)^2} = -\frac{1}{1+x}\Bigg|_0^1 = -\frac{1}{2} + 1 = \frac{1}{2}$$

9. Let A be a set of all 4-digit natural numbers whose exactly one digit is 7. Then the probability that a randomly chosen element of A leaves remainder 2 when divided by 5 is:

(a) $\frac{2}{9}$ (b) $\frac{97}{297}$ (c) $\frac{122}{297}$ (d) $\frac{1}{5}$

Ans. 2

Solution:

Number having exactly one 7 can be

(i) Having 7 at thousand's place = 9^3 = 729

(ii) Not 7 at thousand's place = $3 \times 8 \times 4^2$ = 1944

n(s) = 729 + 1944 = 2673

Favourable cases = having 7 at unit place or having 2 at unit place.

i.e. = $(9\times9) + (8\times9\times2) + (8\times9\times9) = 873$

Required probability = $\frac{873}{2673} = \frac{97}{297}$

10. A plane passes through the points A(1, 2, 3), B(2, 3, 1) and C(2, 4, 2). If O is the origin and P is (2, –1, 1), then the projection of $\overrightarrow{OP}$ on this plane is of length:

(a) $\sqrt{\frac{2}{5}}$ (b) $\sqrt{\frac{2}{7}}$ (c) $\sqrt{\frac{2}{3}}$ (d) $\sqrt{\frac{2}{11}}$

Ans. (d)

Solution:

$\overrightarrow{AB} = \hat{i} + \hat{j} - 2k$

$\overrightarrow{AC} = \hat{i} + 2\hat{j} - k$

Normal to plane $\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & k \\ 1 & 1 & -2 \\ 1 & 2 & -1 \end{vmatrix} = 3\hat{i} - \hat{j} + k$$

$\overrightarrow{OP} = 2\hat{i} - \hat{j} + k$

$$\sin\theta = \frac{\overrightarrow{OP}.\vec{n}}{|\overrightarrow{OP}| \times |\vec{n}|} = \frac{6+1+1}{\sqrt{11}.\sqrt{6}} = \frac{8}{\sqrt{66}}$$

$$\cos\theta = \sqrt{1 - \frac{64}{66}} = \frac{1}{\sqrt{33}}$$

$$\text{Projection} = |\overrightarrow{OP}| \cos\theta = \sqrt{6} \times \frac{1}{\sqrt{33}} = \sqrt{\frac{2}{11}}$$

11. The integral $\int \frac{e^{3\log_e 2x} + 5e^{2\log_e 2x}}{e^{4\log_e x} + 5e^{3\log_e x} - 7e^{2\log_e x}} dx$, x > 0, is equal to:

(a) $4\log_e |x^2 + 5x - 7| + c$ (b) $\log_e |x^2 + 5x - 7| + c$

(c) $\frac{1}{4}\log_e |x^2 + 5x - 7| + c$ (d) $\log_e \sqrt{x^2 + 5x - 7} + c$

Ans. (a)

Solution:

$$I = \int \frac{e^{3\ln 2x} + 5e^{2\ln 2x}}{4^{4\ln x} + 5e^{3\ln x} - 7e^{2\ln x}} dx$$

$$I = \int \frac{(2x)^3 + 5(2x)^2}{x^4 + 5x^3 - 7x^2} dx = \int \frac{8x + 20}{x^2 + 5x - 7} dx$$

Let $x^2 + 5x - 7 = t$

$(2x + 5)dx = dt$

$$I = 4\int \frac{dt}{t} = 4\ln|t| + c$$

$$I = 4\ln|x^2 + 5x - 7| + c$$

12. If for the matrix, A = $\begin{bmatrix} 1 & -\alpha \\ \alpha & \beta \end{bmatrix}$, $AA^T = I_2$, then the value of $\alpha^4 + \beta^4$ is:

(a) 1 (b) 2 (c) 4 (d) 3

Ans. (a)

Solution:

$$A = \begin{bmatrix} 1 & -\alpha \\ \alpha & \beta \end{bmatrix}$$

$\because\ AA^T = I$

So, $I + \alpha^2 = 1 \quad \Rightarrow \quad \alpha^2 = 0$

and $\alpha^2 + \beta^2 = 1 \quad \Rightarrow \quad \beta^2 = 1$

then $\alpha^4 + \beta^4 = 1$

13. The following system of linear equations

2x + 3y + 2z = 9

3x + 2y + 2z = 9

x – y + 4z = 8

(a) has a solution (α, β, γ) satisfying $\alpha + \beta^2 + \gamma^3 = 12$

(b) has a unique solution

(c) does not have any solution

(d) has infinitely many solutions

Ans. (b)

Solution:

Determinant of coefficients of given equations is

$$\begin{vmatrix} 2 & 3 & 2 \\ 3 & 2 & 2 \\ 1 & -1 & 4 \end{vmatrix} = 2(8+2) - 3(12-2) + 2(-3-2)$$

= 20 – 30 – 10 = – 20 $\neq 0$

$\therefore$ Hence the system of equation have unique solution

14. $\operatorname{cosec}\left[2\cot^{-1}(5) + \cos^{-1}\left(\frac{4}{5}\right)\right]$ is equal to:

(a) $\frac{65}{56}$ (b) $\frac{65}{33}$ (c) $\frac{75}{56}$ (d) $\frac{56}{33}$

Ans. (a)

Solution:

$$\operatorname{cosec}\left(\cot^{-1}5 + \cos^{-1}\left(\frac{4}{5}\right)\right)$$

$$\Rightarrow \operatorname{cosec}\left(\tan^{-1}\left(\frac{5}{12}\right) + \tan^{-1}\left(\frac{3}{4}\right)\right)$$

$$\Rightarrow \operatorname{cosec}\left(\tan^{-1}\left(\frac{56}{33}\right)\right)$$

$$\Rightarrow \operatorname{cosec}\left(\operatorname{cosec}^{-1}\left(\frac{65}{56}\right)\right) = \frac{65}{56}$$

15. If $\alpha, \beta \in R$ are such that 1 – 2i (here $i^2 = -1$) is a root of $z^2 + \alpha z + \beta = 0$, then $(\alpha - \beta)$ is equal to:

(a) – 3 (b) – 7 (c) 7 (d) 3

Ans. (b)

Solution:

As $\alpha, \beta \in R$ roots are 1 – 2i and 1 + 2i

$-\alpha = 2 \Rightarrow \alpha = -2$

and $\beta = (1)^2 - (2i)^2 = 5 \Rightarrow \alpha - \beta = -7$

16. The contrapositive of the statement "If you will work, you will earn money" is:

(a) If you will earn money, you will work
(b) You will earn money, if you will not work
(c) If you will not earn money, you will not work
(d) To earn money, you need to work

Ans. (c)

Solution:

Contrapositive of $A \to B$ is $\sim B \to \sim A$

$\therefore$ Contrapositive of the given statement will be ~(you will earn money) → ~(you will work)

i.e., if you will not earn money, you will not work

17. In a group of 400 people, 160 are smokers and non-vegetarian; 100 are smokers and vegetarian and the remaining 140 are non-smokers and vegetarian. Their chances of getting a particular chest disorder are 35%, 20% and 10% respectively. A person is chosen from the group at random and is found to be suffering from the chest disorder. The probability that the selected person is a smoker and non-vegetarian is;

(a) $\frac{7}{45}$ (b) $\frac{28}{45}$ (c) $\frac{14}{45}$ (d) $\frac{8}{45}$

Ans. (b)

Solution:

n(smokers + Non-vegetarian) = 160 = n(A_1)

Let

$\Rightarrow$ P(A)$_1$ = 0.4

n(smokers + vegetarian) = 100 = n(A_2)

similarly P(A_2) = 0.25

n(Non-smokers + vegetarian) = 140 = n(A_3) and P(A_3) = 0.35

Let event E of getting chest disorder i.e.,

P(E/A_1) = 0.35, P(E/A_2) = 0.2, P(E/A_3) = 0.1 to find P(A_1/E)

using Baye's theorem we get

$$P(A_1/E) = \frac{P(E/A_1).P(A_1)}{P(E/A_1).P(A_1) + P(E/A_2)P(A_2) + P(E/A_3).P(A_3)}$$

$$= \frac{0.35 \times 0.4}{(0.35 \times 0.4) + (0.2 \times 0.25) + (0.1 \times 0.35)}$$

$$= \frac{140}{140 + 50 + 35} = \frac{140}{225} = \frac{28}{45}$$

18. If $0 < x, y < \pi$ and $\cos x + \cos y - \cos(x + y) = \frac{3}{2}$, then $\sin x + \cos y$ is equal to:

(a) $\frac{1+\sqrt{3}}{2}$ (b) $\frac{\sqrt{3}}{2}$ (c) $\frac{1}{2}$ (d) $\frac{1-\sqrt{3}}{2}$

Ans. (a)

Solution:

LHS = $\cos x + \cos y - \cos(x + y)$

$= 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) - \left(2\cos^2\frac{x+y}{2} - 1\right)$

$\leq 2\cos\frac{x+y}{2} - 2\cos^2\frac{x+y}{2} + 1$

$\because \left[\frac{x-y}{2} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \Rightarrow 0 < \cos\left(\frac{x-y}{2}\right) \leq 1\right]$

$= 1 - 2\left(\cos^2\left(\frac{x+y}{2}\right) - \cos\left(\frac{x+y}{2}\right)\right)$

$= 1 - 2\left[\left(\cos\left(\frac{x+y}{2}\right) - \frac{1}{2}\right)^2 - \frac{1}{4}\right]$

$= \frac{3}{2} - 2\left(\cos\left(\frac{x+y}{2}\right) - \frac{1}{2}\right)^2 \leq \frac{3}{2}$

But given that LHS = $\frac{3}{2}$

$\therefore \cos\frac{x-y}{2} = 1$ and $\cos\left(\frac{x+y}{2}\right) = \frac{1}{2}$

$\Rightarrow x - y = 0$ and $x + y = \frac{2\pi}{3}$

$\Rightarrow x = y = \frac{\pi}{3}$

$\Rightarrow \sin x + \cos y = \frac{\sqrt{3}+1}{2}$

19. The shortest distance between the line $x - y = 1$ and the curve $x^2 = 2y$ is:

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{2}$ (c) 0 (d) $\frac{1}{2\sqrt{2}}$

Ans. (d)

Solution:

Equation of line parallel to $x - y = 1$ is

$x - y = c$ ...(i)

If line $x - y = c$ is tangent to parabola $x^2 = 2y$ then $x^2 = 2(x - c)$ has unique roots

$x^2 - 2x + 2c = 0$

$\therefore D = 0 \Rightarrow 4 - 4 \times 1 \times 2c = 0$

$\therefore c = \frac{1}{2}$

$\therefore$ Tangent of parabola is $x - y = \frac{1}{2}$

$\therefore$ Shortest distance = $\frac{\left|1 - \frac{1}{2}\right|}{\sqrt{2}} = \frac{1}{2\sqrt{2}}$ units

20. Let A be a 3×3 matrix with det (A) = 4. Let R_i denote the i^{th} row of A. If a matrix B is obtained by performing the operation $R_2 \to 2R_2 + 5R_3$ on 2A, then det(B) is equal to:

(a) 64 (b) 128 (c) 80 (d) 16

Ans. (a)

Solution:

Given det (A) = 4

On application of $R_2 \to 2R_2 + 5R_3$ on 2A we have 2^3. 2 det (A) = $16 \times 4 = 64$

Section-II

Numerical Value Type Questions: This section contains 10 questions. In section II, attempt any five questions out of 10. The answer to each question is a NUMERICAL VALUE. For each question, enter the correct numerical value (in decimal notation, truncated/.rounded-off to the second decimal place; e.g. 06.25, 07.00, -00.33, -00.30, 30.27, -27.30) using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.

1. If the curve, y = y(x) represented by the solution of the differential equation $(2xy^2 – y)\,dx + x\,dy = 0$, passes through the intersection of the lines 2x – 3y = 1 and 3x + 2y = 8, then $|y(1)|$ is equal to ________ .

Ans. 1

Solution:

$\because\ (2xy^2 - y)\,dx + x\,dy = 0$

$$\Rightarrow\ 2x\,dx = \frac{y\,dx - x\,dy}{y^2}$$

$$\Rightarrow\ 2x\,dx = d\left(\frac{x}{y}\right)$$

On integrating both sides we get

$$x^2 = \frac{x}{y} + c \qquad \text{...(i)}$$

The point eq intersection of lines 2x – 3y = 1 and 3x + 2y = 8 is (2, 1)

$\because$ Curve (1) passes through (2, 1) then c = 2

$$\therefore\ y(x) = \frac{x}{x^2 - 2}$$

$$\therefore\ y(1) = \frac{1}{1-2} = -1$$

$$\therefore\ |y(1)| = 1$$

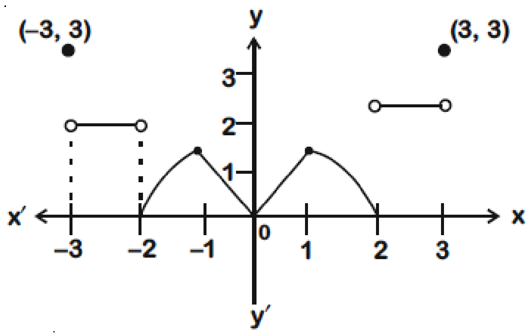
2. A function f is defined on [–3, 3] as $f(x) = \begin{cases} \min\{|x|, 2-x^2\}, & -2 \le x \le 2 \\ [|x|], & 2 < |x| \le 3 \end{cases}$

where [x] denotes the greatest integer $\le x$. The number of points, where f is not differentiable in (–3, 3) is ________ .

Ans. 5

Solution:

$$f(x) = \begin{cases} \min\{|x|, 2-x^2\}, & -2 \le x \le 2 \\ [|x|] & , 2 < |x| \le 3 \end{cases}$$



$$\text{Now, } f(x) = \begin{cases} 3 & , \quad x = -3 \\ 2 & , \quad -3 < x < -2 \\ 2 - x^2 & , \quad 2 \le x < -1 \\ -x & , \quad -1 \le x < 0 \\ x & , \quad 0 \le x \le 2 \\ 2 - x^2 & , \quad 1 \le x \le 2 \\ 2 & , \quad 2 < x < 3 \\ 3 & , \quad x = 3 \end{cases}$$

∴ The points in (–3, 3) where function is not differentiable is x = –2, –1, 0, 1 and 2.

∴ Total number of non-differentiable points = 5

3. The value of $\int_{-2}^{2} |3x^2 - 3x - 6|\, dx$ is ________ .

Ans. 19

Solution:

∵ $3x^2 - 3x - 6 = 3(x^2 - x - 2)$

$= 3(x - 2)(x + 1)$

$\therefore \int_{-2}^{2} |3x^2 - 3x - 6|\, dx$

$= \int_{-2}^{-1} (3x^2 - 3x - 6)\, dx + \int_{-1}^{2} (6 + 3x - 3x^2)\, dx$

$= 3\left\{\int_{-2}^{-1} (x^2 - x - 2)\, dx + \int_{-1}^{2} (2 + x - x^2)\, dx\right\}$

$= 3\left\{\left(\frac{x^3}{3} - \frac{x^2}{2} - 2x\right)_{-2}^{-1} + \left(2x + \frac{x^2}{2} - \frac{x^3}{3}\right)_{-1}^{2}\right\}$

$= 3\left\{\left(-\frac{1}{3} - \frac{1}{2} + 2\right) - \left(-\frac{8}{3} - 2 + 4\right) + \left(4 + 2 - \frac{8}{3}\right) - \left(-2 + \frac{1}{2} + \frac{1}{3}\right)\right\}$

= 19

4. Let $\vec{a}=\hat{i}+\alpha\hat{j}+3\hat{k}$ and $\vec{b}=3\hat{i}-\alpha\hat{j}+\hat{k}$. If the area of the parallelogram whose adjacent sides are represented by the vectors $\vec{a}$ and $\vec{b}$ is $8\sqrt{3}$ square units, then $\vec{a}.\vec{b}$ is equal to ______.

Ans. 2

Solution:

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & \alpha & 3\\ 3 & -\alpha & 1\end{vmatrix}=4\alpha\hat{i}+8\hat{j}-4\alpha\hat{k}=4\left(\alpha\hat{i}+2\hat{j}-\alpha\hat{k}\right)$$

$\because 8\sqrt{3}=4\sqrt{2\alpha^2+4} \quad \Rightarrow \quad \alpha=\pm 2$

$\vec{a}.\vec{b}=3-\alpha^2+3=2$

5. A line 'I' passing through origin is perpendicular to the lines

$l_1:\vec{r}=(3+t)\hat{i}+(-1+2t)\hat{j}+(4+2t)\hat{k}$

$l_2:\vec{r}=(3+2s)\hat{i}+(3+2s)\hat{j}+(2+s)\hat{k}$

If the co-ordinates of the point in the first octant on 'I_2" at a distance of $\sqrt{17}$ from the point of intersection of 'I' and 'I_1' are (a, b, c) then 18(a + b + c) is equal to________ .

Ans. 44

Solution:

$I_1:\dfrac{x-3}{1}=\dfrac{y+1}{2}=\dfrac{z-4}{2}$ and

$I_2:\dfrac{x-3}{2}=\dfrac{y-3}{2}=\dfrac{z-2}{1}$

$$\vec{b_1}\times\vec{b_2}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & 2\\ 2 & 2 & 1\end{vmatrix}=-2\hat{i}+3\hat{j}-2\hat{k}$$

So, $I:\dfrac{x}{-2}=\dfrac{y}{3}=\dfrac{z}{-2}$

Point of intersection of I and I_1 can be considered as

$P(-2\lambda, 3\lambda, -2\lambda)$ and $\dfrac{-2\lambda-3}{1}=\dfrac{3\lambda+1}{2}=\dfrac{-2\lambda-4}{2}$

$\Rightarrow P(2,-3,2)$

Let a point Q on I_2 as $Q(2\mu+3, 2\mu+3, \mu+2)$

$\because PQ=\sqrt{17} \Rightarrow (2\mu+1)^2+(2\mu+6)^2+\mu^2=17$

$\Rightarrow \mu=-\dfrac{10}{9}$ or -2

As Q lies in 1st octant, then $Q\left(\dfrac{7}{9},\dfrac{7}{9},\dfrac{8}{9}\right)$, Hence 18(a + b + c) = 44

6. If the remainder when x is divided by 4 is 3, then the remainder when $(2020 + x)^{2022}$ is divided by 8 is ________ .

Ans. 1

Solution:

$\because$ x = 4y + 3

then $(2020 + x)^{2022} = (2023 + 4y)^{2022}$

$= (4\lambda - 1)^{2022}$

$= (16\lambda^2 - 8\lambda + 1)^{2022}$

$= (8\mu + 1)^{1011}$

$= 8\gamma + 1$ where $\lambda, \mu, \gamma \in N$

7. A line is a common tangent to the circle $(x - 3)^2 + y^2 = 9$ and the parabola $y^2 = 4x$. If the two points of contact (a, b) and (c, d) are distinct and lie in the first quadrant, then 2(a + c) is equal to ________ .

Ans. 9

Solution:

Let equation of tangent to $y^2 = 4x$ as

$$y = mx + \frac{1}{m}$$

If it is a common tangent, then

$$\left|\frac{3m + \frac{1}{m}}{\sqrt{1 + m^2}}\right| = 3 \Rightarrow m = \pm\frac{1}{\sqrt{3}}$$

Equation of common tangent having point of contact in first quadrant; $y = \frac{x + 3}{\sqrt{3}}$.

The tangent intersects the parabola at $\left(3, 2\sqrt{3}\right)$ and circle at $\left(\frac{3}{2}, \frac{3\sqrt{3}}{2}\right)$

So, 2(a + c) = 9

8. The total number of two digit numbers 'n', such that $3^n + 7^n$ is a multiple of 10, is ______ .

Ans. 45

Solution:

$3^n + 7^n$ is divisible by (3 + 7) if n is odd.

So, number of two digit odd numbers = 45

9. If $\lim\limits_{x\to 0}\dfrac{ax-\left(e^{4x}-1\right)}{ax\left(e^{4x}-1\right)}$ exists and is equal to b, then the value of a – 2b is ________ .

Ans. 5

Solution:

$$L=\lim_{x\to 0}\frac{a-\left(\dfrac{e^{4x}-1}{x}\right)}{a\left(e^{4x}-1\right)}$$

$$L=\lim_{x\to 0}\frac{a-\dfrac{1}{x}\left[\dfrac{4x}{\lfloor 1}+\dfrac{(4x)^2}{\lfloor 2}+...\right]}{a\left[\dfrac{4x}{\lfloor 1}+\dfrac{(4x)^2}{\lfloor 2}+...\right]}$$

Clearly, a – 4 = 0 $\Rightarrow a=4$

$$L=\frac{-8}{16}=\frac{-1}{2}=b$$

So, a – 2b = 4 + 1 = 5

10. If the curves $x = y^4$ and $xy = k$ cut at right angles, then $(4k)^6$ is equal to _____

Ans. 4

Solution:

$C_1 : y^4 = x$ and $C_2: xy = k$

Point of intersection of C_1 and C_2 is $\left(k^{\frac{4}{5}},k^{\frac{1}{5}}\right)$

$$m_1=\frac{dy_1}{dx}=\frac{1}{4y^3}=\frac{1}{4x^{\frac{3}{5}}}$$

$$m_2=\frac{dy_2}{dx}=\frac{k}{x^2}=-\frac{1}{k^{\frac{3}{5}}}$$

$$\because\ m_1:m_2=-1\Rightarrow\frac{1}{4k^{\frac{6}{5}}}=1\Rightarrow 4k^{\frac{6}{5}}=1$$

$$\Rightarrow (4k)^6=4$$