JEE (Mains) Mathematics Solution

25.2.2021(Shift-1)

Section-I

Multiple Choice Questions: This section contains 20 multiple choice questions.

Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.

1. Let α be the angle between the lines whose direction cosines satisfy the equations l + m – n = 0 and $l^2 + m^2 - n^2 = 0$. Then the value of $\sin^4\alpha + \cos^4\alpha$ is:

(a) $\frac{3}{4}$ (b) $\frac{5}{8}$ (c) $\frac{1}{2}$ (d) $\frac{3}{8}$

Ans. (b)

Solution:

$l + m - n = 0 \Rightarrow l = n - m$ ...(i)

$l^2 + m^2 - n^2 = 0$

Substitute I from (i) into (ii)

$\Rightarrow$ $(n - m)^2 + m^2 - n^2 = 0$

$2m(m - n) = 0$

$m = 0$ or $m = n$

Case-I

$m = 0 \Rightarrow I = n$

$l^2 + m^2 + n^2 = 1 \Rightarrow l^2 = \frac{1}{2} \Rightarrow l_1, l_2 = \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}$

$l = n \Rightarrow n_1, n_2 = \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}$

DCs $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$ or $\left(\frac{-1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right)$ are DCs of same line $\rightarrow l_1$

Case-II

$m = n \Rightarrow l = 0 \Rightarrow l_1, l_2 = 0$

$l^2 + m^2 + n^2 = 1 \Rightarrow m^2 = \frac{1}{2} \Rightarrow m_1, m_2 = \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}$

$m = n \Rightarrow n_1, n_2 = \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}$

DCs $\left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ or $\left(0, \frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$ are DCs of I_2

$\cos\alpha = l_1l_2 + m_1m_2 + n_1n_2 = 0 + 0 \pm \frac{1}{2} = \pm\frac{1}{2}$

$\cos^2\alpha = \frac{1}{4}, \sin^2\alpha = \frac{3}{4} \Rightarrow \sin^4\alpha + \cos^4\alpha = \frac{5}{8}$

2. If Rolle's theorem holds for the function $f(x) = x^3 - ax^2 + bx - 4,\ x \in [1, 2]$ with $f'\left(\frac{4}{3}\right) = 0$, then ordered pair (a, b) is equal to:

(a) (5, – 8) (b) (5, 8) (c) (–5, –8) (d) (–5, 8)

Ans. (b)

Solution:

$f(x) = x^3 - ax^2 + bx - 4$

$f(1) = f(2)$

$\Rightarrow 3a - b = 7$ ...(i)

$f'(x) = 3x^2 - 2ax + b$

$f'\left(\frac{4}{3}\right) = 0$

$\Rightarrow 8a - 3b = 16$ ...(ii)

(i) and (ii)

$\Rightarrow a = 5, b = 8$

3. All possible values of $\theta \in [0, 2\pi]$ for which $\sin 2\theta + \tan 2\theta > 0$ lie in:

(a) $\left(0, \frac{\pi}{2}\right) \cup \left(\pi, \frac{3\pi}{2}\right)$ (b) $\left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{4}\right) \cup \left(\pi, \frac{7\pi}{6}\right)$

(c) $\left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{4}\right) \cup \left(\frac{3\pi}{2}, \frac{11\pi}{6}\right)$ (d) $\left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{4}\right) \cup \left(\pi, \frac{5\pi}{4}\right) \cup \left(\frac{3\pi}{2}, \frac{7\pi}{4}\right)$

Ans. (d)

Solution:

$$\sin 2\theta + \tan 2\theta > 0 \frac{\sin 2\theta + \cos 2\theta + \sin 2\theta}{\cos 2\theta} > 0 \tan 2\theta(1 + \cos 2\theta) > 0$$

$\Rightarrow \tan 2\theta > 0$ and $\cos 2\theta \neq -1$

$\Rightarrow 2\theta \in \left(n\pi, n\pi + \frac{\pi}{2}\right)$

$\Rightarrow \theta \in \left(\frac{n\pi}{2}, (2n+1)\frac{\pi}{4}\right)$....(i)

$2\theta \neq (2n+1)\pi$

$\theta \neq (2n+1)\frac{\pi}{2}$ ....(ii)

$\Rightarrow \theta \in [0, 2\pi]$

$\therefore \theta \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{4}\right) \cup \left(\pi, \frac{5\pi}{4}\right) \cup \left(\frac{3\pi}{2}, \frac{7\pi}{4}\right)$

4. A man is observing, from the top of a tower, a boat speeding towards the tower from a certain point A, with uniform speed. At the point, angle of depression of the boat with the man's eye is 30^0 (Ignore man's height). After sailing for 20 seconds, towards the base of the tower (which is at the level of water), the boat has reached a point B, where the angle of depression is 45^0. Then the time taken (in seconds) by the boat from B to reach the base of the tower is:

(a) $10\sqrt{3}$ (b) 10 (c) $10\left(\sqrt{3}+1\right)$ (d) $10\left(\sqrt{3}-1\right)$

Ans. (c)

Solution:

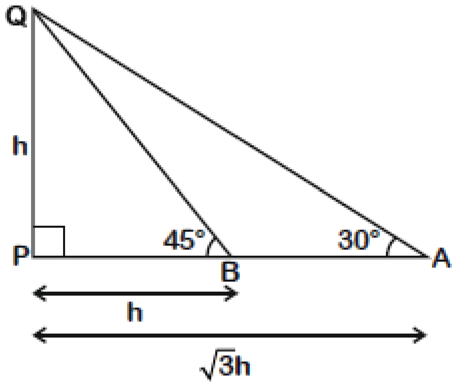
Let PQ = h

PB = h cot 45^0= h

$PA = h\cot 30 = \sqrt{3}h$

AB = PA –PB

$= \left(\sqrt{3}-1\right)h$



$\text{Speed} = \dfrac{\text{Distance}}{\text{Time}}$

$\dfrac{AB}{20} = \dfrac{PB}{t}$

$\dfrac{\left(\sqrt{3}-1\right)h}{20} = \dfrac{h}{t} \Rightarrow t = \dfrac{20}{\sqrt{3}-1} = 10\left(\sqrt{3}+1\right)$

5. The value of $\int_{-1}^{1} x^2 e^{[x^3]} dx$, where [t] denotes the greatest integer $\le t$, is:

(a) $\dfrac{e+1}{3}$ (b) $\dfrac{1}{3e}$ (c) $\dfrac{e+1}{3e}$ (d) $\dfrac{e-1}{3e}$

Ans. (c)

Solution:

$\int_{-1}^{1} x^2 e^{[x^3]} dx = \int_{-1}^{0} x^2 e^{[x^3]} dx + \int_{0}^{1} x^2 e^{[x^3]} dx$

$= \int_{-1}^{0} x^2.e^{-1} dx + \int_{0}^{1} x^2.e^{0} dx$

$= \dfrac{1}{e}\int_{-1}^{0} x^2 dx + \int_{0}^{1} x^2 dx$

$= \dfrac{1}{e}\dfrac{x^3}{3}\Big|_{-1}^{0} + \dfrac{x^3}{3}\Big|_{0}^{1}$

$= \dfrac{1}{3e} + \dfrac{1}{3} = \dfrac{e+1}{3e}$

6. The statement $A \to (B \to A)$ is equivalent to:

(a) $A \to (A \leftrightarrow B)$ (b) $A \to (A \wedge B)$ (c) $A \to (A \to B)$ (d) $A \to (A \vee B)$

Ans. (d)

Solution:

$B \to A = \sim B \vee A$

Also $A \to (B \to A) = A \to (\sim B \vee A) = \sim A \vee (\sim B \vee A)$

$= \sim A \vee \sim B \vee A = \sim A \vee A \vee \sim B = t \vee \sim B = t$

$A \to (A \vee B)$

$= \sim A \vee (A \vee B)$

$= (\sim A \vee A) \vee B$

$= t \vee B = t$

7. Let f, g: $N \to N$ such that f(n + 1) = f(n) + f(1) $\forall n \in N$ and g be any arbitrary function. Which of the following statements is NOT true?

(a) If g is onto, then fog is one-one

(b) If f is onto, then f(n) = $n \forall n \in N$

(c) f is one-one

(d) If fog is one-one, then g is one-one

Ans. (a)

Solution:

Given f, g: $N \to N$

& f(n + 1) = f(n) + 1

$$\left.\begin{array}{l} \Rightarrow f(2) = 2f(1) \\ \Rightarrow f(3) = 3f(1) \\ \quad f(4) = 4f(4) \\ \quad \ldots\ldots\ldots\ldots \\ \quad f(n) = nf(1) \end{array}\right\} \Rightarrow f \text{ is one} - \text{one.}$$

Now if f is onto $\Rightarrow f(1) = 1$

$\Rightarrow \boxed{f(n) = n}$

Also it is clear if fog is one-one $\Rightarrow$ g will be one-one.

So only option (1) is not correct.

8. The total number of positive integral solutions (x, y, z) such that xyz = 24 is:

(a) 36 (b) 30 (c) 45 (d) 24

Ans. (b)

Solution:

Given xyz = 24 = $2^3 \times 3$

So total number of positive integral solutions (x, y, z)

$= {}^{3+3-1}C_{3-1} \times {}^{1+3-1}C_{3-1}$

$= {}^5C_2 \times {}^3C_2$

$= 10 \times 3$

= 30

9. When a missile is fired a ship, the probability that it is intercepted is $\frac{1}{3}$ and the probability that the missile hits the target, given that it is not intercepted, is $\frac{3}{4}$. If three missiles are fired independently from the ship, then the probability that all three hit the target, is:

(a) $\frac{1}{27}$ (b) $\frac{3}{8}$ (c) $\frac{3}{4}$ (d) $\frac{1}{8}$

Ans. (d)

Solution:

Given P(when it is intercepted) = $\frac{1}{3}$

$\Rightarrow$ P(being not intercepted) = $1 - \frac{1}{3} = \frac{2}{3}$ & also when it not intercepted, probability it hits the target = $\frac{3}{4}$

So when such 3 missiles launched then P(all 3 hitting the target)

$= \left(\frac{2}{3} \times \frac{3}{4}\right) \times \left(\frac{2}{3} \times \frac{3}{4}\right) \times \left(\frac{2}{3} \times \frac{3}{4}\right)$

$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$

$= \frac{1}{8}$

10. Let the lines $(2-i)z=(2+i)\bar{z}$ and $(2+i)z+(i-2)\bar{z}-4i=0$, (here $i^2=-1$)

be normal to a circle C. If the line $iz+\bar{z}+1+i=0$ is tangent to this circle C, then its radius is:

(a) $3\sqrt{2}$ (b) $\frac{3}{\sqrt{2}}$ (c) $\frac{3}{2\sqrt{2}}$ (d) $\frac{1}{2\sqrt{2}}$

Ans. (c)

Solution:

Given lines are

$(2-i)z=(2+i)\bar{z}$ ....(1)

and $(2+i)z+(i-2)\bar{z}-4i=0$

or $-i(2+i)z-i(i-2)\bar{z}-4=0$

$\Rightarrow (1-2i)z+(1+2i)\bar{z}-4=0$ ....(2)

Let z = x + iy

So from (1) we get the line $y=\frac{x}{2}$ ....(3)

and from (2) (1 – 2i) (x + iy) + (1 + 2i) (x – iy) – 4 = 0

$\Rightarrow x+2y-2=0$ ....(4)

On solving (3) and (4) we get x = 1, $y=\frac{1}{2}$

These lines were normal to the circle.

So centre = $\left(1,\frac{1}{2}\right)$

Now the line $iz+\bar{z}+1+i=0$

or $i(1-i)z+(1-i)\bar{z}+(1+1)=0$

$\Rightarrow (1+i)z+(1-i)\bar{z}+2=0$

$\Rightarrow (z+\bar{z})+i(z-\bar{z})+2=0 \quad \Rightarrow \quad 2x-2y+2=0$

x – y + 1 = 0

$\because$ This line is tangent to circle

So, $r=\frac{\left|1-\frac{1}{2}+1\right|}{\sqrt{1+1}}=\frac{\left|\frac{3}{2}\right|}{\sqrt{2}}$

$r=\frac{3}{2\sqrt{2}}$

11. If the curves, $\frac{x^2}{a}+\frac{y^2}{b}=1$ and $\frac{x^2}{c}+\frac{y^2}{d}=1$ intersect each other at an angle of 90^0, then which of the following relations is TRUE?

(a) $a - c = b + d$ (b) $a + b = c + d$ (c) $a - b = c - d$ (d) $ab = \frac{c+d}{a+b}$

Ans. (c)

Solution:

Given, curves are $\frac{x^2}{a}+\frac{y^2}{b}=1$ [Ellipse]

and other curves can be written as

$\frac{x^2}{c}-\frac{y^2}{(-d)}=1,$ Which is a hyperbola

Since these both are orthogonal

So, $\sqrt{a-b}=\sqrt{c-d}$

$\Rightarrow a-b=c-d$

12. The image of the point (3, 5) in the line $x - y + 1 = 0$, lies on:

(a) $(x-4)^2+(y+2)^2=16$ (b) $(x-4)^2+(y-4)^2=8$

(c) $(x-2)^2+(y-2)^2=12$ (d) $(x-2)^2+(y-4)^2=4$

Ans. (d)

Solution:

Given the point (3, 5)

and the line $x - y + 1 = 0$

So, let the image is (x, y)

So, we have

$$\frac{x-3}{1}=\frac{y-5}{-1}=-\frac{2(3-5+1)}{1+1}$$

$\Rightarrow x=4, y=4$

$\Rightarrow$ Point (4, 4)

Which will satisfy the curve

$(x-2)^2+(y-4)^2=4$

as $(4-2)^2+(4-4)^2$

$= 4 + 0 = 4$

13. The value of the integral

$$\int \frac{\sin\theta.\sin 2\theta\left(\sin^6\theta+\sin^4\theta+\sin^2\theta\right)\sqrt{2\sin^4\theta+3\sin^2\theta+6}}{1-\cos 2\theta}\,d\theta \text{ is:}$$

(where c is a constant of integration)

(a) $\frac{1}{18}\left[9-2\cos^6\theta-3\cos^4\theta-6\cos^2\theta\right]^{\frac{3}{2}}+c$ (b) $\frac{1}{18}\left[11-18\cos^2\theta+9\cos^4\theta-2\cos^6\theta\right]^{\frac{3}{2}}+c$

(c) $\frac{1}{18}\left[9-2\sin^6\theta-3\sin^4\theta-6\sin^2\theta\right]^{\frac{3}{2}}+c$ (d) $\frac{1}{18}\left[11-18\sin^2\theta+9\sin^4\theta-2\sin^6\theta\right]^{\frac{3}{2}}+c$

Ans. (b)

Solution:

$$\int \frac{\sin\theta(2\sin\theta)\cos\theta\left(\sin^6\theta+\sin^4\theta+\sin^2\theta\right)\sqrt{2\sin^4\theta+3\sin^2\theta+6}}{1-\left(1-2\sin^2\theta\right)}d\theta$$

Put $\sin\theta=t$

$\Rightarrow \cos\theta\, d\theta=dt$

$$\Rightarrow \int \frac{t^2\left(t^6+t^4+t^2\right)\sqrt{2t^4+3t^2+6}}{t^2}\,dt$$

$$\int\left(t^5+t^3+t\right)\sqrt{2t^6+3t^4+6t^2}\,dt$$

Put $2t^6 + 3t^4 + 6t^2 = k$

$\Rightarrow 12\left(t^5+t^3+t\right)dt=dk$

$$\Rightarrow \frac{1}{12}\int\sqrt{k}\,dk$$

$$\Rightarrow \frac{2k^{\frac{3}{2}}}{12.3}$$

$$\Rightarrow \frac{1}{18}\left(2\sin^6\theta+3\sin^4+6\sin^2\right)^{\frac{3}{2}}+C$$

$$= \frac{1}{18}\left(11-18\cos^2\theta+9\cos^4\theta-2\sin^6\theta\right)^{\frac{3}{2}}+C$$

14. If a curve passes through the origin and the slope of the tangent to it at any point (x, y) is $\dfrac{x^2-4x+y+8}{x-2}$, then this curve also passes through the point:

(a) (5, 5) (b) (4, 5) (c) (4, 4) (d) (5, 4)

Ans. (a)

Solution:

$$\frac{dy}{dx}=\frac{x^2-4x+y+8}{x-2}=\frac{(x-2)^2+(y+4)}{(x-2)}$$

Put x – 2 = t

$\Rightarrow dx=dt$

$$\Rightarrow \frac{dy}{dt}=\frac{t^2+y+4}{t}$$

$$\Rightarrow \frac{dy}{dt}-\frac{y}{t}=t+\frac{4}{t}$$

I.F. = $e^{-\int\frac{1}{t}dt}=\frac{1}{t}$

$$\Rightarrow \frac{y}{t}=t-\frac{4}{t}+C$$

$$y=(x-2)^2-4+C(x-2)$$

$\downarrow$ (0, 0)

C = 0

y = $(x-2)^2$ – 4 also passes through (5, 5)

15. If $0<\theta,\phi<\dfrac{\pi}{2}$, $x=\sum\limits_{n=0}^{\infty}\cos^{2n}\theta$, $y=\sum\limits_{n=0}^{\infty}\sin^{2n}\phi$ and $z=\sum\limits_{n=0}^{\infty}\cos^{2n}\theta.\sin^{2n}\phi$ then:

(a) xyz = 4 (b) xy – z = (x + y) z (c) xy + yz + zx = z (d) xy + z = (x + y) z

Ans. (d)

Solution:

$$x=\sum_{n=0}^{\infty}\cos^{2n}\theta=1+\cos^2\theta+\cos^4\theta+...=\frac{1}{1-\cos^2\theta}=\operatorname{cosec}^2\theta$$

$$y=\sum_{n=0}^{\infty}\sin^{2n}\phi=1+\sin^2\phi+\sin^4\phi+...=\frac{1}{1-\sin^2\phi}=\sec^2\phi$$

$$z=\sum_{n=0}^{\infty}\cos^{2n}\theta\sin^{2n}\phi$$

$$=1+\cos^2\theta\sin^2\phi+\left(\cos^2\theta\sin^2\phi\right)^2+...=\frac{1}{1-\cos^2\theta\sin^2\phi}$$

$$\Rightarrow z=\frac{1}{1-\left(1-\frac{1}{x}\right)\left(1-\frac{1}{y}\right)}$$

$$\Rightarrow 1-1+\frac{1}{x}+\frac{1}{y}-\frac{1}{xy}=\frac{1}{z}$$

$$\Rightarrow \frac{x+y}{xy}=\frac{z+xy}{xyz}$$

$$\Rightarrow (x+y)z=xy+z$$

16. The equation of the line through the point (0, 1, 2) and perpendicular to the line $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-1}{-2}$ is:

(a) $\frac{x}{3}=\frac{y-1}{4}=\frac{z-2}{-3}$ (b) $\frac{x}{3}=\frac{y-1}{4}=\frac{z-2}{3}$ (c) $\frac{x}{-3}=\frac{y-1}{4}=\frac{z-2}{3}$ (d) $\frac{x}{3}=\frac{y-1}{-4}=\frac{z-2}{3}$

Ans. (c)

Solution:

Let equation of line $\frac{x}{a}=\frac{y-1}{b}=\frac{z-2}{c}$

for being perpendicular to $\frac{x}{2}=\frac{y+1}{3}=\frac{z-1}{-2}$

we get 2a + 3b – 2c = 0

Hence satisfying line is $\frac{x-1}{-3}=\frac{y-1}{4}=\frac{z-2}{3}$

17. $\lim_{n\to\infty}\left(1+\frac{1+\frac{1}{2}+......+\frac{1}{n}}{n^2}\right)^n$ is equal to:

(a) $\frac{1}{2}$ (b) 0 (c) 1 (d) $\frac{1}{e}$

Ans. (c)

Solution:

$$L=\lim_{n\to\infty}\left(\frac{1+\left(1+\frac{1}{2}+\frac{1}{3}+....\frac{1}{n}\right)}{n^2}\right)^n$$

if $n\to\infty\ 1+\frac{1}{2}+\frac{1}{3}+.....+\frac{1}{n}<n$

hence $\lim_{n\to\infty}\frac{1+\frac{1}{2}+...+\frac{1}{n}}{n^2}=0$

L is of 1^∞ form

$$L=e^{\lim_{n\to\infty}\left(\frac{1+\frac{1}{2}+\frac{1}{3}+....+\frac{1}{n}}{n^2}\right).n}=e^o=1$$

18. The coefficients a, b and c of the quadratic equation $ax^2 + bx + c = 0$ are obtained by throwing a dice three times. The probability that this equation has equal roots is:

(a) $\frac{1}{54}$ (b) $\frac{1}{36}$ (c) $\frac{5}{216}$ (d) $\frac{1}{72}$

Ans. (c)

Solution:

For equal roots $b^2 = 4ac$

$a, b, c \in \{1, 2, 3, 4, 5, 6\}$

Favourable cases

$b = 2 \quad a = c = 1$

$b = 4 \quad (a, c) = (1, 4), (4, 1)$ and $(2, 2)$

$b = 6 \quad (a, c) = (3, 3)$

Total possible ordered triplets

$(a, b, c) = 6^3 – 216$

Favourable cases = 5

$\therefore$ Required probability = $\frac{5}{16}$

19. A tangent is drawn to the parabola $y^2 = 6x$ which is perpendicular to the line $2x + y = 1$. Which of the following points does NOT lie on it?

(a) (0, 3) (b) (–6,0) (c) (4, 5) (d) (5, 4)

Ans. (d)

Solution:

Slope of line : $2x + y = 1$ is – 2

Slope of line perpendicular to given line is $\frac{1}{2}$

$\therefore$ Equation of tangents to parabola $y^2 = 6x$ is

$$y = \frac{1}{2}x + \frac{\frac{6}{4}}{\frac{1}{2}}$$

$$y = \frac{1}{2}x + 3$$

$x – 2y + 6 = 0$

$\therefore$ (5, 4) does not lies on $x – 2y + 6 = 0$

20. The integer 'k', for which the inequality $x^2 - 2(3k - 1)x + 8k^2 - 7 > 0$ is valid for every x in R, is:

(a) 2 (b) 3 (c) 4 (d) 0

Ans. (b)

Solution:

$x^2 - 2(3x - 1)x + 8k^2 - 7 > 0, \ \forall x \in R$

Here $D < 0$

$4(3k - 1)^2 - 4.1.(8k^2 - 7) < 0$

$9k^2 - 6k + 1 - 8k^2 + 7 < 0$

$k^2 - 6k + 8 < 0$

$(k - 2)(k - 4) < 0$

$k \in (2, 4)$

Section-II

Numerical Value Type Questions: This section contains 10 questions. In section II, attempt any five questions out of 10. The answer to each question is a NUMERICAL VALUE. For each question, enter the correct numerical value (in decimal notation, truncated/.rounded-off to the second decimal place; e.g. 06.25, 07.00, -00.33, -00.30, 30.27, -27.30) using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.

1. Let $A_1, A_2, A_3, ...$ be squares such that for each $n \geq 1$, the length of the side of A_n equals the length of diagonal of A_{n+1}. If the length of A_1 is 12 cm, then the smallest value of n for which area of A_n is less than one, is ___________ .

Ans. 9

Solution:

$A_1 = 12$, Let side of square 2 be A_2

Given diagonal of A_{n+1} = Side of A_n

$\Rightarrow 2A_2^2 = A_1^2 \Rightarrow A_2 = A_{1/\sqrt{2}} \qquad \left(\text{i.e., } A_{n+1} = \frac{A_n}{\sqrt{2}}\right)$

$\Rightarrow A_2 = \frac{A_1}{\sqrt{2}}, A_3 = \frac{A_2}{\sqrt{2}} = \frac{A_1}{2} ...$

$A_{n+1} = (\sqrt{2}.\sqrt{2}...(n-1) \text{ times})^{-1} A_1$

$\text{Area} = (A_{n+1})^2 = \frac{A_1^2}{2^{(n-1)}} < 1$

$144 < 2^{n-1} \Rightarrow n - 1 \geq 8$

n = 9

2. The number of points, at which the function $f(x)=|2x+1|-3|x+2|+|x^2+x-2|, x\in R$ is not differentiable is________ .

Ans. 2

Solution:

$f(x)=|2x+1|-3|x+2|+|x^2+x-2|$

$=|2x+1|-3|x+2|+|(x+2)(x-1)|$

$$\therefore\ f(x)=\begin{cases} x^2+2x+3 & x<-2 \\ -x^2-6x-5 & -2\le x<-\frac{1}{2} \\ -x^2-2x-3 & \frac{-1}{2}\le x<1 \\ x^2-7 & 1\le x \end{cases}$$

at x = - 2 f(x) is continuous,

LHD = –2 & RHD = –2 Hence differentiable

at x = $\frac{-1}{2}$ f(x) is continuous,

LHD = –5 & RHD = – 1 Hence non-differentiable at x = 1 f(x) is continuous,

LHD = –4 & RHD = 2 Hence non-differentiable

$\therefore$ f(x) is non-differentiable at x = $\frac{1}{2}$ and 1

3. The total number of numbers, lying between 100 and 1000 than can be formed with the digits 1, 2, 3, 4, 5, if the repetition of digits is not allowed and numbers are divisible by either 3 or 5, is ____________ .

Ans. 32

Solution:

The numbers are lying between 100 and 1000 then each number is of three digits.

The possible combination of 3 digits numbers are

1, 2, 3; 1, 2, 4; 1, 2, 5; 1, 3, 4; 1, 3, 5; 1, 4, 5; 2, 3, 4; 2, 3, 5; 2, 4, 5; and 3, 4, 5.

The numbers which are divisible by 3 are 1, 2, 3; 3, 4, 5; 1, 3, 5 and 2, 3, 4.

$\therefore$ Total number of numbers = $4\times 3!=24$

The number divisible by 5 are 1, 2, 5; 2, 3, 5; 1, 4, 5 and 2, 4, 5.

$\therefore$ Number divisible by 5 = $4\times 2!=8$

$\therefore$ Total required number = 24 + 8 = 32

4. Let f(x) be a polynomial of degree 6 in x, in which the coefficient of x^6 is unity and it has extrema at x = – 1, and x = 1. If $\lim_{x\to 0}\frac{f(x)}{x^3}=1$, then 5. f(2) is equal to _________.

Ans. 144

Solution:

Let f(x) = $x^6 + ax^5 + bx^3 + cx^3 + dx^2 + ex + f$

$\therefore\ \lim_{x\to 0}\frac{f(x)}{x^3}=1 \Rightarrow d=e=f=0 \text{ and } c=1$

So, f(x) = $x^6 + ax^5 + bx^4 + x^3$

f′(x) = $6x^5 + 5ax^4 + 4bx^3 + 3x^2$

$\because$ f'(1) = 0 = f'(−1)

$\Rightarrow 5a + 4b = -9$ and $5a - 4b = 3$

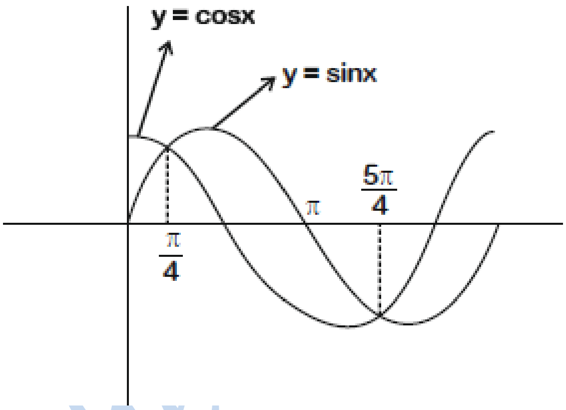
$\Rightarrow a = -\frac{3}{5}$ and $b = -\frac{3}{2}$

Then 5. f(2) = $5\left[2^6 - \frac{3}{5}.2^5 - \frac{3}{2}.2^4 + 2^3\right]$ = 144

5. The graphs of sine and cosine functions, intersect each other at a number of points and between two consecutive points of intersection, the two graphs enclose the same area A. Then A^4 is equal to __________.

Ans. 64

Solution:



$$A = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x - \cos x)dx = -\cos x - \sin x \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$$

$\Rightarrow A = \sqrt{2} + \sqrt{2} = 2\sqrt{2}$

$\Rightarrow A^4 = 64$

6. If A = $\begin{bmatrix} 0 & -\tan\left(\frac{\theta}{2}\right) \\ \tan\left(\frac{\theta}{2}\right) & 0 \end{bmatrix}$ and $(I_2, + A)(I_2 - A)^{-1} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, then $13(a^2 + b^2)$ is equal to____ .

Ans. 13

Solution:

$$I_2 + A = \begin{bmatrix} 1 & -\tan\frac{\theta}{2} \\ \tan\frac{\theta}{2} & 1 \end{bmatrix} \quad ....(1)$$

$$I_2 - A = \begin{bmatrix} 1 & \tan\frac{\theta}{2} \\ -\tan\frac{\theta}{2} & 1 \end{bmatrix}$$

$$\Rightarrow (I_2 - A)^{-1} = \frac{1}{\sec^2\frac{\theta}{2}} \begin{bmatrix} 1 & -\tan\frac{\theta}{2} \\ \tan\frac{\theta}{2} & 1 \end{bmatrix} \quad ....(2)$$

$$= \frac{1}{\sec^2\frac{\theta}{2}} \begin{bmatrix} 1 - \tan^2\frac{\theta}{2} & -2\tan\frac{\theta}{2} \\ 2\tan\frac{\theta}{2} & 1 - \tan^2\frac{\theta}{2} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Clearly a = $\cos\theta$ and b = $\sin\theta$, then $(a^2 + b^2)$ = 13

7. The locus of the point of intersection of the lines $\left(\sqrt{3}\right)kx+ky-4\sqrt{3}=0$ and $\sqrt{3}x-y-4\left(\sqrt{3}\right)k=0$ is a conic, whose eccentricity is __________ .

Ans. 2

Solution:

$$L_1:\sqrt{3x}+y=\frac{4\sqrt{3}}{k}$$

and $L_2:\sqrt{3x}-y=4\sqrt{3k}$

So point of intersection with always satisfy

$$\left(\sqrt{3x}-y\right)\left(\sqrt{3x}-y\right)=48$$

$$\Rightarrow \frac{x^2}{16}-\frac{y^2}{48}=1$$

$$e=\sqrt{1+\frac{48}{16}}=2$$

8. If the system of equations

kx + y + 2z = 1

3x– y – 2z = 2

–2x – 2y – 4z = 3

has infinitely many solutions, then k is equal to ________ .

Ans. 21

Solution:

kx + y + 2z = 1 ...(1)

–3x + y + 2z = – 2 ...(2)

$x+y+2z=\frac{-3}{2}$ ...(3)

from (2) and (3) we get

$$x=\frac{1}{8} \text{ and } y+2z=-\frac{13}{8}$$

Substituting these values in (1) we get k = 21

9. Let $A = \begin{bmatrix} x & y & z \\ y & z & x \\ z & x & y \end{bmatrix}$, where x, y and z are real numbers such that x + y + z > 0 and xyz = 2.

If $A^2 = I_3$, then the value of $x^3 + y^3 + z^3$ is ________ .

Ans. 7

Solution:

$\because\ A^2 = I_3 \Rightarrow x^2 + y^2 + z^2 = 1$ and $xy + yz + zx = 0$ then x + y + z = 1

$\because\ |A| = 3xyz - x^3 - y^3 - z^3$

$= -(x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$

$\Rightarrow 6 - (x^3 + y^3 + z^3) = -1$

$\Rightarrow x^3 + y^3 + z^3 = 7$

We will not get the real numbers x, y, z satisfying these conditions.

10. Let $\vec{a} = \hat{i} + 2\hat{j} - k$, $\vec{b} = \hat{i} - \hat{j}$ and $\vec{c} = \hat{i} - \hat{j} - k$ be three given vectors, if $\vec{r}$ is a vector such that $\vec{r} \times \vec{a} = \vec{c} \times \vec{a}$ and $\vec{r}.\vec{b} = 0$, then $\vec{r}.\vec{a}$ is equal to ___________.

Ans. 12

Solution:

$\because\ \vec{a}.\vec{b} = -1, \vec{b}.\vec{c} = 2, \vec{c}.\vec{a} = 0$

$\vec{r} \times \vec{a} = \vec{c} \times \vec{a} \Rightarrow (\vec{r} \times \vec{a}) \times \vec{b} = (\vec{c} \times \vec{a}) \times \vec{b}$

$\Rightarrow (\vec{r}.\vec{b})\vec{a} - (\vec{a}.\vec{b})\vec{r} = (\vec{b}.\vec{c})\vec{a} - (\vec{a}.\vec{b})\vec{c}$

$\Rightarrow \vec{r} = 2\vec{a} + \vec{c}$

then $\vec{r}.\vec{a} = 2|\vec{a}|^2 + \vec{a}.\vec{c}$

= 12