JEE (Mains) Mathematics Solution

24.2.2021 (Shift-1)

Section-I

Multiple Choice Questions: This section contains 20 multiple choice questions. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.

1. If $\int \frac{\cos x - \sin x}{\sqrt{8 - \sin 2x}} dx = a \sin^{-1}\left(\frac{\sin x + \cos x}{b}\right) + c$, where c is a constant of integration, then the ordered pair (a, b) is equal to:

(a) (−1, 3) (b) (3, 1) (c) (1, – 3) (d) (1, 3)

Ans. (d)

Solution:

$$\int \frac{\cos x - \sin x}{\sqrt{8 - \sin 2x}} dx = \int \frac{\cos x - \sin x}{\sqrt{9 - (\sin x + \cos x)^2}} dx$$

let sin x + cos x = t

(cos x – sin x) dx = dt

$= \int \frac{dt}{\sqrt{9 - t^2}}$

$= \sin^{-1}\left(\frac{t}{3}\right) + c$

$= \sin^{-1}\left(\frac{\sin x + \cos x}{3}\right) + c$

Hence (a, b) = (1, 3)

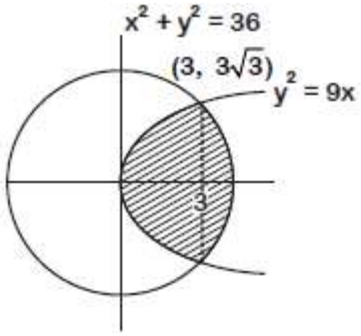
2. The area (in sq. units) of the part of the circle $x^2 + y^2 = 36$, which is outside the parabola $y^2 = 9x$, is:

(a) $12\pi + 3\sqrt{3}$ (b) $24\pi + 3\sqrt{3}$ (c) $12\pi - 3\sqrt{3}$ (d) $24\pi - 3\sqrt{3}$

Ans. (d)

Solution:

Area of the shaded region



$= 2\left[\int_0^3 3\sqrt{x}dx + \int_3^6 \sqrt{36 - x^2}dx\right]$

$= 2\left[2x^{3/2}\Big|_0^3 + \left(\frac{1}{2}x\sqrt{36 - x^2} + 18\sin^{-1}\left(\frac{x}{6}\right)\right)\Big|_3^6\right]$

$= 2\left[6\sqrt{3} + 9\pi - \frac{9\sqrt{3}}{2} - 3\pi\right] = 3\sqrt{3} + 12\pi$

Required area = $36\pi - \left(3\sqrt{3} + 12\pi\right)$ = $24\pi - 3\sqrt{3}$

3. If $e^{\left(\cos^2 x+\cos^4 x+\cos^6 x+...\infty\right)\log_e 2}$ satisfies the equation $t^2 - 9t + 8 = 0$, then the value of $\frac{2\sin x}{\sin x+\sqrt{3}\cos x}\left(0<x<\frac{\pi}{2}\right)$ is:

(a) $\frac{3}{2}$ (b) $\sqrt{3}$ (c) $2\sqrt{3}$ (d) $\frac{1}{2}$

Ans. (d)

Solution:

$e^{\left(\frac{\cos^2 x}{1-\cos^2 x}\right)\ln 2}=2^{\cot^2 x}$

$\because\ t=1$ or 8

So, $2^{\cot^2 x}=2^0$ or $2^3 \Rightarrow \cot^2 x=0$ or 3

$\because\ x\in\left(0,\frac{\pi}{2}\right)$ then $\cot x=\sqrt{3}\Rightarrow x=\frac{\pi}{6}$

$$\frac{2\sin x}{\sin x+\sqrt{3}\cos x}=\frac{2\left(\frac{1}{2}\right)}{\frac{1}{2}+\sqrt{3}.\left(\frac{\sqrt{3}}{2}\right)}=\frac{1}{2}$$

4. The population P = P(t) at time 't' of a certain species follows the differential equation $\frac{dP}{dt}=0.5P-450$. If P(0) = 850, then the time at which population becomes zero is:

(a) $\log_e 18$ (b) $\frac{1}{2}\log_e 18$ (c) $\log_e 9$ (d) $2\log_e 18$

Ans. (d)

Solution:

$\because\ \frac{dP}{dt}=\frac{1}{2}(P-900)$

$\Rightarrow \frac{dP}{P-900}=\frac{1}{2}dt \Rightarrow \ln|P-900|=\frac{1}{2}t+c$

When t = 0, P = 850 $\Rightarrow\ c=\ln 50$

When P = 0, t = 2(ln 900 – ln 50) = 2ln 18

5. The statement among the following that is a tautology is:

(a) $A\vee(A\wedge B)$ (b) $B\rightarrow[A\wedge(A\rightarrow B)]$

(c) $[A\wedge(A\rightarrow B)]\rightarrow B$ (d) $A\wedge(A\vee B)$

Ans. (c)

Solution:

(a) $A\vee(A\wedge B)=A$

(b) $\because\ A\wedge(A\rightarrow B)=A\wedge(\sim A\vee B)=A\wedge B$

So, $B\rightarrow(A\wedge B)=\sim B\vee(A\wedge B)=\sim B\vee A$

(c) $(A\wedge(A\rightarrow B))\rightarrow B=(A\wedge B)\rightarrow B=\sim(A\wedge B)$

$\vee B=\sim A\vee\sim B\vee B$ (Tautology)

(d) $A\wedge(A\vee B)=A$

6. Let p and q be two positive numbers such that $p + q = 2$ and $p^4 + q^4 = 272$.
Then p and q are roots of the equation:

(a) $x^2 - 2x + 2 = 0$ (b) $x^2 - 2x + 8 = 0$
(c) $x^2 - 2x + 136 = 0$ (d) $x^2 - 2x + 16 = 0$

Ans. (d)

Solution:

$\because\ p^4 + q^4 = (p+4)^4 - 4pq(p^2+q^2) - 6p^2q^2$

$\Rightarrow 272 = 16 - 4pq(4 - 2pq) - 6p^2q^2$

$2p^2q^2 - 16pq - 256 = 0$

$\Rightarrow pq = -8$ or 16

$\because$ p, q > 0, so pq = 16

Required quadratic equation is

$x^2 - 2x + 16 = 0$

7. The system of linear equations

$3x - 2y - kz = 10$

$2x - 4y - 2z = 6$

$x + 2y - z = 5m$

is inconsistent if:

(a) $k \neq 3, m \neq \frac{4}{5}$ (b) $k \neq 3, m \in R$ (c) $k = 3, m = \frac{4}{5}$ (d) $k = 3, m \neq \frac{4}{5}$

Ans. (d)

Solution:

Here $\Delta = \begin{vmatrix} 3 & -2 & -k \\ 2 & -4 & -2 \\ 1 & 2 & -1 \end{vmatrix} = 0$

$\Rightarrow 3(4 + 4) + 2(-2 + 2) - k(k + 4) = 0$

$\Rightarrow 24 + 0 - 8k = 0 \quad \Rightarrow \quad \boxed{k = 3}$

Now,

$\Delta_1 = \begin{vmatrix} 10 & -2 & -3 \\ 6 & -4 & -2 \\ 5m & 2 & -1 \end{vmatrix} = 10(4+4) + 2(-6+10m) - 3(12+20m)$

$= 80 - 12 + 20m - 36 - 60m = 32 - 40m$

$\Delta_2 = \begin{vmatrix} 3 & 10 & -3 \\ 2 & 6 & -2 \\ 1 & 5m & -1 \end{vmatrix} = 3(-6+10m) - 10(-2+2) - 3(10m-6)$

$= -18 + 30m + 0 - 30m + 18 = 0$

$\Delta_3 = \begin{vmatrix} 3 & -2 & 10 \\ 2 & -4 & 6 \\ 1 & 2 & 5m \end{vmatrix} = 3(-20m-12) + 2(10m-6) + 10(4+4)$

$= -60m - 36 + 20m - 12 + 80$

$= -40m + 32$

For inconsistent we have k = 3, & $32 - 40m \neq 0 \Rightarrow m \neq \frac{4}{5}$

8. A scientific committee is to be formed from 6 Indians and 8 foreigners, which includes at least 2 Indians and double the number of foreigners as Indians**.** Then the number of ways, the committee can be formed, is:

(a) 560 (b) 1050 (c) 1625 (d) 575

Ans. (c)

Solution:

Indians = 6, Foreigners = 8

According to questions

The no. of ways to form the committee are

(2I, 4F) or (3I, 6F) or (4I, 8F)

$\Rightarrow {}^6C_2 \times {}^8C_4 + {}^6C_3 \times {}^8C_6 + {}^6C_4 \times {}^8C_8$

$= 15 \times 70 + 20 \times 28 + 15 \times 1 = 1625$

9. The equation of the plane passing through the point (1, 2, – 3) and perpendicular to the planes $3x + y - 2z = 5$ and $2x - 5y - z = 7$, is:

(a) $3x - 10y - 2z + 11 = 0$ (b) $6x - y - 2z - 2 = 0$

(c) $6x - 5y + 2z + 10 = 0$ (d) $11x + y + 17z + 38 = 0$

Ans. (d)

Solution:

The given planes are $3x + y - 2z = 5$ ....(1)

$2x - 5y - z = 7$ ....(2)

Since the required plane passes through (1, 2, – 3)

So equation of this plane is

$a(x - 1) + b(y - 2) + c(z + 3) = 0$ ....(3)

Now this plane (3) is $\perp$ to the planes (1) & (2)

So, $3a + b - 2c = 0$ & $2a - 5b - c = 0$

$\Rightarrow \frac{a}{-11} = \frac{b}{-1} = \frac{c}{-17}$

So equation of plane is $11(x - 1) + (y - 2) + 17(2 + 3) = 0$

$\Rightarrow 11x + y + 17z + 38 = 0$

10. If the tangent to the curve $y = x^3$ at the point $P(t, t^3)$ meets the curve again at Q, then the ordinate of the point which divides PQ internally in the ratio 1 : 2 is:

(a) $-2t^3$ (b) $2t^3$ (c) 0 (d) $-t^3$

Ans. (a)

Solution:

Curve is $y = x^3$ ....(1)

So equation of tangent at (t, t^3)

$(y - t^3) = 3t^2(x - t)$ ...(2)

$\because$ It meets again the curve at Q

So solving (1) & (2) we get

$x = -2t \Rightarrow Q = (-2t, -8t^3)$

Now by section formula ordinate = $\frac{2t^3 - 8t^3}{1+2}$

$= \frac{-6t^3}{3}$

$= -2t^3$

11. Let f: $R \to R$ be defined as f(x) = 2x – 1 and g : R – {1} – R be defined as g(x) = $\dfrac{x-\frac{1}{2}}{x-1}$.

Then the composition function f(g(x)) is:

(a) neither one-one nor onto (b) onto but not one-one

(c) both one-one and onto (d) one-one but not onto

Ans. (d)

Solution:

Here f: $R \to R,\ f(x) = 2x - 1$

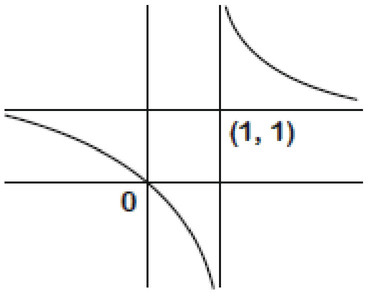
and g: $R - \{1\} \to R\ g(x) = \dfrac{x-\frac{1}{2}}{x-1}$

So, f(g(x)) = 2 g(x) – 1

$= 2\left(\dfrac{x-\frac{1}{2}}{x-1}\right) - 1$

$= \dfrac{2x-1-x+1}{x-1} = \dfrac{x-1+1}{x-1}$

$= 1 + \dfrac{1}{x-1}$



So clearly it is one-one but not onto

12. $\lim\limits_{x\to 0} \dfrac{\int_0^{x^2} \left(\sin\sqrt{t}\right)dt}{x^3}$ is equal to:

(a) 0 (b) $\dfrac{1}{15}$ (c) $\dfrac{2}{3}$ (d) $\dfrac{3}{2}$

Ans. (c)

Solution:

$\lim\limits_{x\to 0} \dfrac{\int_0^{x^2} \left(\sin\sqrt{t}\right)dt}{x^3} = \dfrac{0}{0}\,(\text{form})$

$\Rightarrow$ By D, L Hospital rule

$\lim\limits_{x\to 0} \dfrac{2x \sin x}{3x^2} = \dfrac{2}{3} \lim\limits_{x\to 0}\left(\dfrac{\sin x}{x}\right)$

$= \dfrac{2}{3} \times 1 = \dfrac{2}{3}$

13. The ordinary dice is rolled for a certain number of times. If the probability of getting an odd number 2 times is equal to the probability of getting an even number 3 times, then the probability of getting an odd number for odd number of times is:

(a) $\frac{3}{16}$ (b) $\frac{1}{2}$ (c) $\frac{1}{32}$ (d) $\frac{5}{16}$

Ans. (b)

Solution:

Let number of trials be 'n'

given

$${}^{n}C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^{n-2} = {}^{n}C_3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^{n-3}$$

$\Rightarrow$ n = 5

Probability of getting odd number for odd number of times is

$$= \left(5C_1 + 5C_3 + 5C_5\right)\frac{1}{2^5}$$

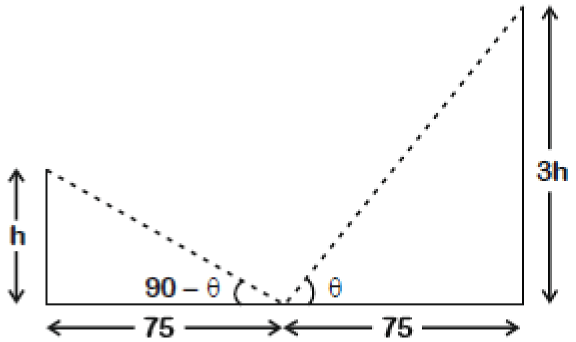
$$= \frac{2^4}{2^5} = \frac{1}{2}$$

14. Two vertical poles are 150m apart and the height of one is three times that of the other. If from the middle point of the line joining their feet, an observed finds the angles of elevation of their tops to be complementary, then the height of the shorter pole (in meters) is:

(a) $25\sqrt{3}$ (b) 30 (c) 25 (d) $20\sqrt{3}$

Ans. (a)

Solution:



Given $\tan\theta = \frac{3h}{75}$ ....(i)

and $\tan(90-\theta) = \frac{h}{75}$ ....(ii)

$\Rightarrow$ Multiplying (i) and (ii) we get,

$$1 = \frac{3h^2}{(75)^2}$$

$\Rightarrow h = 25\sqrt{3}$

15. If f: $R \to R$ is a function defined by f(x) = [x – 1] cos $\left(\frac{2x-1}{2}\right)\pi$, where [.] denotes the greatest integer function, then f is:

(a) discontinuous only at x = 1
(b) continuous for every real x
(c) discontinuous at all integral values of x except at x = 1
(d) continuous only at x = 1

Ans. (b)

Solution:

$$f(x) = [x-1]\cos\left(\frac{2x-1}{2}\right)\pi$$

at x = 1

$$\lim_{x\to 1^-}[x-1]\cos\left(\frac{2x-1}{2}\right)\pi = 0$$

$$\lim_{x\to 1^+}[x-1]\cos\left(\frac{2x-1}{2}\right)\pi = 0$$

f(1) = 0

at any general integer x = k

$$\lim_{x\to K^-}[x-1]\cos\left(\frac{2k-1}{2}\right)\pi = 0$$

$$\lim_{x\to K^+}[x-1]\cos\left(\frac{2k-1}{2}\right)\pi = 0$$

f(k) = 0

$\therefore$ f(x) is continuous $\forall x \in R$

16. The distance of the point (1, 1, 9) from the point of intersection of the line $\frac{x-3}{1} = \frac{y-4}{2} = \frac{z-5}{2}$ and the plane x + y + z = 17 is:

(a) 38 (b) $19\sqrt{2}$ (c) $2\sqrt{19}$ (d) $\sqrt{38}$

Ans. (d)

Solution:

Let a point $P(\lambda)$ on the line

$$\Rightarrow \frac{x-3}{1} = \frac{y-4}{2} = \frac{z-5}{2} = \lambda$$

$$\therefore P(\lambda+3, 2\lambda+4, 2\lambda+5)$$

as P also satisfies the given plane

$$\lambda+3+2\lambda+4+2\lambda+5 = 17$$

$$\Rightarrow 5\lambda = 5 \Rightarrow \lambda = 1$$

$$\therefore P \equiv (4, 6, 7)$$

Distance from (1, 1, 9) is

$$\sqrt{(4-1)^2+(6-1)^2+(7-9)^2}$$

$$= \sqrt{9+25+4} = \sqrt{38}$$

17. The locus of the mid-point of the line segment joining the focus of the parabola $y^2 = 4ax$ to a moving point of the parabola, is another parabola whose directrix is

(a) $x = a$ (b) $x = \frac{a}{2}$ (c) $x = 0$ (d) $x = -\frac{a}{2}$

Ans. (c)

Solution:

Let the moving point be $P(at^2, 2at)$

Focus of given parabola is (a, 0)

Let point of required locus (h, k)

$\therefore \frac{at^2 + a}{2} = h$ ...(i)

and $\frac{2at + 0}{2} = k$ ...(ii)

$\Rightarrow \frac{a}{2}(t^2 + 1) = h$ ...(iii)

and $t = \frac{k}{a}$ ...(iv)

By (iii) and (iv) we have

$\frac{a}{2}\left(\frac{k^2}{a^2} + 1\right) = h$

Locus of $k^2 + a^2 = 2ah$

$\Rightarrow y^2 = 2a\left(x - \frac{a}{2}\right)$

Equation of directrix $x - \frac{a}{2} + \frac{a}{2} = 0$

$\Rightarrow x = 0$

18. A man is walking on a straight line. The arithmetic mean of the reciprocals of the intercepts of this line on the coordinate axes is $\frac{1}{4}$. Three stones A, B and C are placed at the points (1, 1), (2, 2) and (4, 4) respectively. Then which of these stones is/are on the path of the man?

(a) C only (b) B only (c) All the three (d) A only

Ans. (b)

Solution:

Let line be $\frac{x}{a} + \frac{y}{b} = 1$ ...(i)

given $\frac{\frac{1}{a} + \frac{1}{b}}{2} = \frac{1}{4}$

$\Rightarrow \frac{1}{a} + \frac{1}{b} = \frac{1}{2}$ ...(ii)

By (i) and (ii), we get $\frac{x}{a} + \left(\frac{1}{2} - \frac{1}{a}\right)y = 1$

$\Rightarrow \lambda(x - y) + \left(\frac{y}{2} - 1\right) = 0$

$\therefore$ Represents family of line passing through (2, 2)

19. The function

$$f(x)=\frac{4x^3-3x^2}{6}-2\sin x+(2x-1)\cos x:$$

(a) Decreases in $\left[\frac{1}{2},\infty\right)$ (b) Decreases in $\left(-\infty,\frac{1}{2}\right]$

(c) Increases in $\left(-\infty,\frac{1}{2}\right]$ (d) Increases in $\left[\frac{1}{2},\infty\right)$

Ans. (d)

Solution:

$\because\ f(x)=\frac{4x^3-3x^2}{6}-2\sin x+(2x-1)\cos x$

On differentiating both sides w. r. t. x we get f′(x) = $\frac{12x^2-6x}{6}-2\cos x-(2x-1)\sin x+2\cos x$

$f'(x)=2x^2-x-(2x-1)\sin x$

$f'(x)=(2x-1)(x-\sin x)$

When $x>\frac{1}{2}, 2x-1>0$ and $x-\sin x>0.$

$\therefore\ f'(x)>0$ if $x>\frac{1}{2}$

$f(x)$ is increasing in $\left[\frac{1}{2},\infty\right)$

20. The value of

$-{}^{15}C_1 + 2.\ {}^{15}C_2 - 3.\ {}^{15}C_3 + \ldots. -15.\ {}^{15}C_{15} + {}^{14}C_1 + {}^{14}C_3 + {}^{14}C_5 + \ldots + {}^{14}C_{11}$ is

(a) 2^{14} (b) $2^{16}-1$ (c) $2^{13}-13$ (d) $2^{13}-14$

Ans. (d)

Solution:

$\because\quad (1+x)^{14}={}^{14}C_0+{}^{14}C_1x+{}^{14}C_2x^2+\ldots+{}^{14}C_{14}x^{14}$ ...(i)

$\therefore\quad 2^{14}={}^{14}C_0+{}^{14}C_1+{}^{14}C_2+\ldots.+{}^{14}C_{14}$ ...(ii)

$0={}^{14}C_0-{}^{14}C_1+{}^{14}C_2\ldots+{}^{14}C_{14}$ ...(iii)

$\therefore\quad {}^{14}C_1+{}^{14}C_3+\ldots.+{}^{14}C_{13}=2^{13}$ ...(iv)

and $(1-x)^{15} = {}^{15}C_0 - {}^{15}C_1x + {}^{15}C_2x^2 + \ldots{}^{15}C_{15}x^{15}$

Differentiate w. r. t. x we get

$-15(1-x)^{14} = -{}^{15}C_1 + 2.\ {}^{15}C_2x + \ldots.-15.\ {}^{15}C_{15}x^{14}$

Put x = 1, we get

$-{}^{15}C_1 + 2.\ {}^{15}C_2 - 3.\ {}^{15}C_3 + \ldots. -15.{}^{15}C_{15} = 0$ ...(v)

From equation (iv) + equation (v) we get

$-{}^{15}C_1 + 2.\ {}^{15}C_2 + \ldots.-15{}^{15}C_{15} + {}^{14}C_1 + {}^{14}C_r + \ldots + {}^{14}C_{11}$

$= 2^{13} - {}^{14}C_{13}$

$= 2^{13} - 14$

Section-II

Numerical Value Type Questions: This section contains 10 questions. In section II, attempt any five questions out of 10. The answer to each question is a NUMERICAL VALUE. For each question, enter the correct numerical value (in decimal notation, truncated/.rounded-off to the second decimal place; e.g. 06.25, 07.00, -00.33, -00.30, 30.27, -27.30) using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.

1. Let $A = \{n \in N : n \text{ is a 3-digit number}\}$

$B = \{9k + 2 : k \in N\}$

and $C = \{9k + I : k \in N\}$ for some I $(0 < I < 9)$

If the sum of all the elements of the set $A \cap (B \cup C)$ is 274×400, then I is equal to ______.

Ans. 5

Solution:

Sum of all elements of $A \cap (B \cup C)$ is

$$274 \times 400 = \sum_{k=0}^{99} \{(99 + 9k + I) + (99 + 9k + 2)\}$$

$$\Rightarrow 274 \times 400 = 200 \times 100 + 100I + 18.\left(\frac{99 \times 100}{2}\right)$$

$$\Rightarrow 274 \times 4 = 200 + I + 9 \times 99$$

$$\Rightarrow I = 5$$

2. If $\int_{-a}^{a} (|x| + |x-2|)dx = 22, (a > 2)$ and [x] denotes the greatest integer $\le x$, then $\int_{a}^{a} (x + [x])dx$ is equal to

________ .

Ans. 3

Solution:

$$\int_{-a}^{a} (|x| + |x-2|)\, dx = \frac{1}{2}a^2 + \frac{1}{2}a^2 + \frac{1}{2}(a-2)^2 + \frac{1}{2}(a+2)^2$$

$$\Rightarrow 22 = 2a^2 + 4 \Rightarrow a = 3$$

Now,

$$\int_{3}^{-3} (x + [x]dx = -\int_{-3}^{3} [x]dx = -\int_{-3}^{-2} [x]dx = -\int_{-3}^{-2} (-3)dx$$

3. Let $P = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & \alpha \\ 3 & -5 & 0 \end{bmatrix}$, where $\alpha \in R$. Suppose $Q = [a_{ij}]$ is a matrix satisfying PQ = KI_3 for some

non-zero $k \in R$. If $q_{23} = -\frac{k}{8}$ and $|Q| = \frac{k^2}{2}$, then $\alpha^2 + k^2$ is equal to ________ .

Ans. 17

Solution:

$$\because\ Q = k.P^{-1} \text{ and } |P||Q| = k^3, |Q| = \frac{k^2}{2} \text{ then } |P| = 2k$$

$$\because\ q_{23} = \frac{kC_{32}}{|P|} \text{ (Where } C_{ij} \text{ is co-factor of } P_{ij} \text{ of P)}$$

$-\frac{k}{8}=-\frac{(3\alpha+4)k}{2k}\Rightarrow 3\alpha+4=\frac{k}{4}$ ...(1)

Also $|P|=2k \Rightarrow 12\alpha+20=2k$

$\Rightarrow k=6\alpha+10$ ...(2)

From (1) and (2) we get

k = 4 and $\alpha=-1$

then $k^2+\alpha^2=17$

4. If one of the diameters of the circle $x^2 + y^2 - 2x - 6y + 6 = 0$ is a chord of another circle 'C', whose center is at (2, 1), then its radius is ________ .

Ans. 3

Solution:

Circle $x^2 + y^2 - 2x - 6y + 6 = 0$ has centre O, (1, 3) and radius $r_1 = 2$.

Let centre $O_2(2, 1)$ of required circle and its radius being R.

So $R^2=O_1O_2^2=r^2$

$\Rightarrow$ $R^2=5+4$

$\Rightarrow$ R = 3

5. Let B_i (i = 1, 2, 3) be three independent events in a sample space. The probability that only B_1 occurs is α, only B_2 occurs is β and only B_3 occurs is γ. Let p be the probability that none of the events B_i occurs and these 4 probabilities satisfy the equations $(\alpha-2\beta)p=\alpha\beta$ and $(\beta-3\gamma)\ p=2\beta\gamma$ (All the probabilities are assumed to lie in the interval (0, 1)). Then $\frac{P(B_1)}{P(B_3)}$ is equal to ______ .

Ans. 6

Solution:

Let $P(B_1) = x$, $P(B_2) = y$, $P(B_3) = z$

$\alpha=P\left(B_1\cap \bar{B}_2\cap \bar{B}_3\right)=P(B_1)P\left(\bar{B}_2\right)P\left(\bar{B}_3\right)$

$\Rightarrow$ $\alpha=x(1-y)(1-z)$ ...(i)

Similarly $\beta=(1-x)y(1-z)$ ...(ii)

$\gamma=(1-x)(1-y)z$ ...(iii)

$p=(1-x)(1-y)(1-z)$ ...(iv)

(i) & (iv) $\Rightarrow \frac{x}{1-x}=\frac{\alpha}{p}\Rightarrow x=\frac{\alpha}{\alpha+p}$

(iii) & (iv) $\Rightarrow \frac{z}{1-z}=\frac{\gamma}{p}\Rightarrow z=\frac{\gamma}{\gamma+p}$

$\frac{P(B_1)}{P(B_3)}=\frac{x}{z}=\frac{\frac{\alpha}{\alpha+p}}{\frac{\gamma}{\gamma+p}}=\frac{\frac{\gamma+p}{\gamma}}{\frac{\alpha+p}{\alpha}}=\frac{1+\frac{p}{\gamma}}{1+\frac{p}{\alpha}}$ ...(v)

Given that,

$(\alpha-2\beta)\ p=\alpha\beta\Rightarrow \alpha\beta=(\alpha+2\beta)\beta$ ...(vi)

$(\beta-3\gamma)p=2\beta\gamma\Rightarrow 3\gamma p=(p-2\gamma)\beta$ ...(vii)

(vi) & (vii) $\Rightarrow \frac{\alpha}{3\gamma} = \frac{\alpha + 2p}{p - 2\gamma}$

$\Rightarrow p\alpha - 6p\gamma = 5\gamma\alpha$

$\frac{p}{\gamma} - \frac{6p}{\alpha} = 5$

$\frac{p}{\gamma} + 1 = 6\left(\frac{p}{\alpha} + 1\right)$ ...(viii)

(v) & (viii) $\Rightarrow \frac{P(B_1)}{P(B_3)} = 6$

6. If the least and the largest real values of α, for which the equation $z + \alpha|z-1| + 2i = 0 \left(z \in C \text{ and } i = \sqrt{-1}\right)$ has a solution, are p and q respectively, then $4\left(p^2 + q^2\right)$ is equal to ___________

Ans. 10

Solution:

Let z = x + iy, $x, \gamma \in R$

$\because\ z + \alpha|z-1| + 2i = 0.$

$\Rightarrow\ x + i(y+2) + \alpha\sqrt{(x-1)^2 + y^2} = 0.$

$\therefore\ y = -2 \text{ and } x + \alpha\sqrt{(x-1)^2 + 4} = 0.$

Now $\alpha\sqrt{(x-1)^2 + 4} = -x$ *

Squaring both sides : $x^2 = \alpha^2\left((x-1)^2 + 4\right)$

$\therefore\ \left(\alpha^2 - 1\right)x^2 - 2\alpha^2 x + 5\alpha^2 = 0.$

For real x, $D \geq 0.$

$4\alpha^4 - 4.5\alpha^2\left(\alpha^2 - 1\right) \geq 0.$

$\alpha^2\left(5 - 4\alpha^4\right) \geq 0.$

$\therefore\ \alpha \in \left[\frac{-\sqrt{5}}{2}, \frac{\sqrt{5}}{2}\right]$

$\therefore\ \alpha_{min} = \frac{-\sqrt{5}}{2} - p \text{ and } \alpha_{max.} = \frac{\sqrt{5}}{2} = q$

$\Rightarrow 4\left(p^2 + q^2\right) = 10.$

*The step is not permissible as squaring may include extraneous roots.

No maximum value of α exists as is shown in the below procedure.

Let z = x + iy

$x + \alpha\sqrt{(x-1)^2 + y^2} + i(y+2) = 0$

$\Rightarrow\ y = -2 \text{ and } \alpha = \frac{-x}{\sqrt{x^2 - 2x + 5}}$

$$\frac{d\alpha}{dx} = \frac{\left(x^2 - x\right) - \left(x^2 - 2x + 5\right)}{\left(x^2 - 2x + 5\right)^{\frac{3}{2}}} = \frac{x - 5}{\left(x^2 - 2x + 5\right)^{\frac{3}{2}}}$$

So α is decreasing in $(-\infty, 5)$ and increasing in $(5, \infty)$

$$\alpha_{min} = -\frac{5}{\sqrt{20}} = 1\sqrt{\frac{5}{4}} = p(\text{at } x = 5)$$

and $\alpha_{max} = \lim\limits_{x \to -\infty} \dfrac{-x}{\sqrt{x^2 - 2x + 5}} = 1\ q = 1$ (however this value is not achievable.)

7. The minimum value of α for which the equation $\dfrac{4}{\sin x} + \dfrac{1}{1 - \sin x} = \alpha$ has at least one solution in $\left(0, \dfrac{\pi}{2}\right)$ is____ .

Ans. 9

Solution:

Let f(x) = $\dfrac{4}{\sin x} + \dfrac{1}{1 - \sin x}$ where $\sin x \in (0, 1)$

$$\because \frac{\frac{2}{\sin x} + \frac{2}{\sin x} + \frac{1}{1 - \sin x}}{3} \ge \frac{3}{\frac{\sin x}{2} + \frac{\sin x}{2} + 1 - \sin x}$$

$\Rightarrow f(x) \ge 9$

So least value of α is 9.

8. Let M be any 3×3 matrix with entries from the set (0, 1, 2). The maximum number of such matrices, for which the sum of diagonal elements of $M^T M$ is seven is ________ .

Ans. 540

Solution:

Let $\{a_{ij}\}_{3 \times 3}$

$$T_r\left(M^T.M\right) = \sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}^2 = 7$$

So there will be two cases.

Case I: Any seven a_{ij}s are 1 and remaining two elements are zero.

Number of such matrices M = $\dfrac{\lfloor 9}{\lfloor 7 \lfloor 2} = 36$

Case II: Any one elements is 2, any three elements are 1 and reaming elements are 0.

Number of such matrices = $\dfrac{\lfloor 9}{\lfloor 1 \lfloor 3 \lfloor 5} = 504$

Total number of possible matrices M = 540.

9. $\lim_{n\to\infty} \tan\left\{\sum_{r=1}^{n} \tan^{-1}\left(\frac{1}{1+r+r^2}\right)\right\}$ is equal to ________.

Ans. 1

Solution:

$$\tan^{-1}\left(\frac{1}{1+r+r^2}\right) = \tan^{-1}\left(\frac{(r+1)-r}{1+(r+1)r}\right)$$

$= \tan^{-1}(r + 1) - \tan^{-1} r$

So $\sum_{r=1}^{n} \tan^{-1}\left(\frac{1}{1+r+r^2}\right) = \left(\tan^{-1} 2 - \tan^{-1} 1\right)$

$+ (\tan^{-1}3 - \tan^{-1}2) + \ldots\ldots + (\tan^{-1}(n + 1) - \tan^{-1}n)$

$= \tan^{-1}(n + 1) - \tan^{-1}1$

$\Rightarrow \lim_{n\to\infty} \tan\left\{\sum_{r=1}^{n} \tan^{-1}\left(\frac{1}{1+r+r^2}\right)\right\}$

$\lim_{n\to\infty} \tan\left(\tan^{-1}(n+1) - \tan^{-1} 1\right)$

$= \tan\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = 1$

10. Let three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ be such that $\vec{c}$ is coplanar with $\vec{a}$ and $\vec{b}$, $\vec{a}.\vec{c} = 7$ and $\vec{b}$ is perpendicular to $\vec{c}$, where $\vec{a} = -\hat{i} + \hat{j} + k$ and $\vec{b} = 2\hat{i} + k$, then the value of $2\left|\vec{a} + \vec{b} + \vec{c}\right|^2$ is ____.

Ans. 75

Solution:

Let $\vec{c} = \lambda\vec{a} + \mu\vec{b}$ $\left(\because \vec{a}, \vec{b} \text{ and } \vec{c} \text{ are coplanar}\right)$

$\because\ \vec{a}.\vec{c} = 7,\ \vec{b}.\vec{c} = \vec{0}$ and $\vec{a}.\vec{b} = -1$

So $7 = \lambda\left|\vec{a}\right|^2 + \mu\vec{a}.\vec{b} \Rightarrow 3\lambda - \mu = 7$

and $0 = \lambda\vec{a}.\vec{b} + \mu\left|\vec{b}\right|^2 \Rightarrow -\lambda + 5\mu = 0$

Clearly $\lambda = \frac{5}{2}$ and $\mu = \frac{1}{2} \Rightarrow \vec{c} = \frac{5\vec{a} + \vec{b}}{2}$

So $2\left|\vec{a} + \vec{b} + \vec{c}\right|^2 = 2\left|\frac{7\vec{a} + 3\vec{b}}{2}\right|^2$

$= \frac{1}{2}\left|-\hat{i} + 7\hat{j} + 10k\right|^2$

$= \frac{1 + 49 + 100}{2} = 75$