JEE (Mains) Mathematics Solution

26.2.2021 (Shift-2)

Section-I
Multiple Choice Questions: This section contains 20 multiple choice questions. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.

1. If the mirror image of the point (1, 3, 5) with respect to the plane 4x – 5y + 2z = 8 is (α, β, γ), then $5(\alpha+\beta+\gamma)$ equals:

(a) 43 (b) 47 (c) 41 (d) 39

Ans. (b)

Solution:

$$\frac{\alpha-1}{4}=\frac{\beta-3}{-5}=\frac{\gamma-5}{2}=-2\frac{(4\times1-5\times3+2\times5-8)}{16+25+4}$$

$$\frac{\alpha-1}{4}=\frac{\beta-3}{-5}=\frac{\gamma-5}{2}=\frac{2}{5}$$

$$\alpha=\frac{8}{5}+1, \beta=\frac{-10}{5}+3, \gamma=\frac{4}{5}+5$$

$$5|\alpha+\beta+\gamma|=|5\alpha+5\beta+5\gamma|=47$$

2. A natural number has prime factorization given by $n = 2^x3^y5^z$, where y and z are such that y + z = 5 and $y^{-1}+z^{-1}=\frac{5}{6}, y>z.$ Then the number of odd divisors of n, including 1, is:

(a) 12 (b) 6x (c) 11 (d) 6

Ans. 1

Solution:

y + z = 5 ....(i)

$\frac{1}{y}+\frac{1}{z}=\frac{y+z}{yz}=\frac{5}{6}\Rightarrow yz=6$ ....(ii)

Equation with y and z as roots is

$x^2 - 5x + 6 = 0$

x = 2, 3, y = 3, z = 2 (y > z)

$n = 2^x. 3^3. 5^2$

For odd divisors x = 1 only

No. of odd divisors $= 1\times4\times3=12$

3. Let A = {1, 2, 3, ..., 10} and 10} and f: $A \to A$ be defined as $f(k) = \begin{cases} k+1 & \text{if k is odd} \\ k & \text{if k is even} \end{cases}$

Then the number of possible functions g: $A \to A$ such that gof = f is:

(a) 5^5 (b) 10^5 (c) 5! (d) ${}^{10}C_5$

Ans. (b)

Solution:

Note that f(1) = f(2) = 2

f(3) = f(4) = 4

f(5) = f(6) = 6

f(7) = f(8) = 8

f(9) = f(10) = 10

$gof(1) = f(1) \Rightarrow g(2) = f(1) = 2$

$gof(2) = f(2) \Rightarrow g(2) = f(2) = 2$

$gof(3) = f(3) \Rightarrow g(4) = f(3) = 4$

In function(x), 2, 4, 6, 8, 10 should be mapped to 2, 4, 6, 8, 10 respectively. Each of remaining elements can be mapped to any of 10 elements. Number of possible g(x) is 10^5

4. Let slope of the tangent line to a curve at any point P(x, y) be given by $\frac{xy^2+y}{x}$. If the curve intersects the line x + 2y = 4 at x = – 2, then the value of y, for which the point (3, y) lies on the curve, is:

(a) $\frac{18}{35}$ (b) $-\frac{4}{3}$ (c) $-\frac{18}{11}$ (d) $-\frac{18}{19}$

Ans. (d)

Solution:

$$\frac{dy}{dx} = y^2 + \frac{y}{x}$$

$$\frac{dy}{dx} - \frac{y}{x} = y^2$$

$$\frac{1}{y^2}\frac{dy}{dx} - \frac{1}{x} \times \frac{1}{y} = 1$$

Let $\frac{1}{y} = z$

$$\frac{-1}{y^2}\frac{dy}{dx} = \frac{dz}{dx}$$

$$\frac{-dz}{dx} - \frac{1}{x}z = 1$$

$$\frac{dz}{dx} + \frac{1}{x}z = -1$$

$$\text{IF} = e^{\int \frac{1}{x}dx} = e^{\ell n x} = x$$

$$z.x = \int -1.x\, dx$$

$$z.\, x = \frac{-x^2}{2} + c$$

$$\frac{x}{y} = \frac{-x^2}{2} + c \qquad \text{...(i)}$$

Putting x = – 2 in x + 2y = 4, we get y = 3

Put (–2, 3) in (i)

$\Rightarrow c = \frac{4}{3}$

(i) $\Rightarrow \frac{x}{y} = \frac{-x^2}{2} + \frac{4}{3}$ ...(ii)

Put x = 3 in (ii)

$\frac{3}{y} = \frac{-9}{2} + \frac{4}{3}$

$y = \frac{-18}{19}$

5. The triangle of maximum area that can be inscribed in a given circle of radius 'r' is:

(a) An isosceles triangle with base equal to 2r.

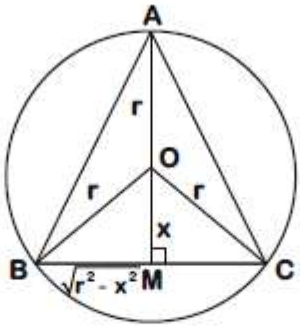
(b) An equilateral triangle of height $\frac{2r}{3}$.

(c) A right angle triangle having two of its sides of length 2r and r.

(d) An equilateral triangle having each of its side of length $\sqrt{3}r$.

Ans. (d)

Solution:



Area of triangle ABC

$A = \frac{1}{2} \times BC \times AM$

$= \frac{1}{2} \times 2\sqrt{r^2 - x^2} \times (r + x)$

$A = (r + x)\sqrt{r^2 - x^2}$

$\frac{dA}{dx} = \sqrt{r^2 - x^2} - \frac{x}{\sqrt{r^2 - x^2}} \times (r + x) = \frac{r^2 - x^2 - rx - x^2}{\sqrt{r^2 - x^2}}$

$= \frac{r^2 - rx - 2x^2}{\sqrt{r^2 - x^2}} = \frac{-(x + r)(2x - r)}{\sqrt{r^2 - x^2}}$

$\frac{dA}{dx} = 0 \Rightarrow x = \frac{r}{2}$

Sign change of $\frac{dA}{dx}$ at $x = \frac{r}{2} \Rightarrow$ A has maximum at x = $\frac{r}{2}$ $BC = 2\sqrt{r^2 - x^2} = \sqrt{3}r$, $AM = \frac{3}{2}r$

$\Rightarrow AB = AC = \sqrt{3}\,r$

6. A seven digit number is formed using digits 3, 3, 4, 4, 4, 5, 5. The probability, that number so formed is divisible by 2, is;

(a) $\frac{1}{7}$ (b) $\frac{6}{7}$ (c) $\frac{4}{7}$ (d) $\frac{3}{7}$

Ans. (d)

Solution:

For even number, units place should be filled with 4 only

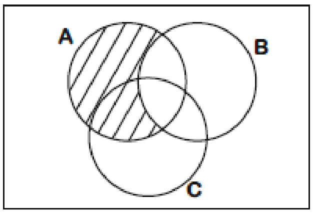
$$P = \frac{\frac{6!}{2!2!2!}}{\frac{7!}{2!3!2!}} = \frac{6!}{2!} \times \frac{3!}{7!} = \frac{3}{7}$$

7. Let $F_1(A, B, C) = (A \cap \sim B) \vee [\sim C \wedge (A \vee B)] \vee \sim A$ and $F_2(A, B) = (A \vee B) \vee (B \rightarrow \sim A)$ be two logical expressions. Then:

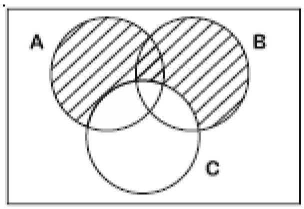
(a) F_1 and F_2 both are tautologies
(b) Both F_1 and F_2 are not tautologies
(c) F_1 is a tautology but F_2 is not a tautology
(d) F_1 is not a tautology but F_2 is a tautology

Ans. (d)

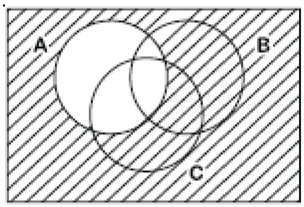
Solution:



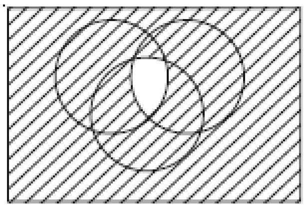
$A \wedge \sim B$



$\sim C \wedge (A \vee B)$



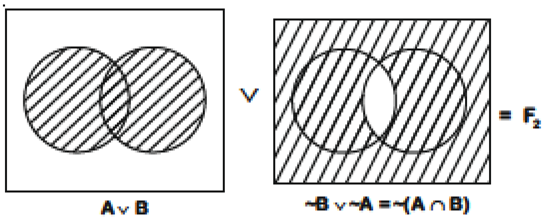
$\sim A$



$\Rightarrow$ F1 is not a tautology

F1

$B \rightarrow \sim A = \sim B \vee \sim A$



$\Rightarrow F_2$ is a tautology

8. Let f: $R \to R$ be defined as $f(x)=\begin{cases} 2\sin\left(-\frac{\pi x}{2}\right), \text{ if } x<-1 \\ |ax^2+x+b|, \text{ if } -1\le x\le 1 \\ \sin(\pi x), \quad \text{if } x>1 \end{cases}$

If f(x) is continuous on R, then a + b equals:

(a) – 1 (b) –3 (c) 3 (d) 1

Ans. (a)

Solution:

$f(-1^-)=2$

$f(-1^+)=|a+b-1|$

$|a+b-1|=2$ ...(i)

$f(1^-)=|a+b+1|$

$f(1^+)=0$

$|a+b+1|=0 \Rightarrow a+b+1=0$

$\Rightarrow a+b=-1$ ...(ii)

9. Let L be a line obtained from the intersection of two planes x + 2y + z = 6 and y + 2z = 4. If point $P(\alpha,\beta,\gamma)$ is the foot of perpendicular from (3, 2, 1) on L, then the value of 21 $(\alpha+\beta+\gamma)$ equals:

(a) 68 (b) 102 (c) 142 (d) 136

Ans. (b)

Solution:

Direction of line L = $\begin{vmatrix} \hat{i} & \hat{j} & k \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 3\hat{i}-2\hat{j}+k$

d. r's = < 3, –2, 1 >

A point on line (–2, 4, 0)

Line = $\frac{x+2}{3}=\frac{y-4}{-2}=\frac{z}{1}$

Foot of perpendicular from (3, 2, 1) be $(3\lambda-2, -2\lambda+4, \lambda)$

$(3\lambda-5).3+(-2\lambda+2)(-2)+(\lambda-1)1=0$

$9\lambda-15+4\lambda-4+\lambda-1=0$

$14\lambda-20=0 \Rightarrow \lambda=\frac{10}{7}$

$(\alpha,\beta,\gamma)=\left(\frac{16}{7},\frac{8}{7},\frac{10}{7}\right)$

$\therefore\ 21(\alpha+\beta+\gamma)=(16+8+10)3=102$

10. Let f(x) be a differentiable function at x = a with f'(a) = 2 and f(a) = 4. Then $\lim\limits_{x\to a}\dfrac{xf(a)-af(x)}{x-a}$ equals:

(a) 4 – 2a (b) a + 4 (c) 2a – 4 (d) 2a + 4

Ans. (a)

Solution:

$$L=\lim_{x\to a}\frac{xf(a)-af(x)}{x-a}\qquad\left[\frac{0}{0}\text{form}\right]$$

Using L'Hospital rule we get

$$L=\lim_{x\to a}\frac{f(a)-af'(x)}{1}$$

f(a) – af '(a) = 4 – 2a

11. Let A(1, 4) and B(1, – 5) be two points. Let P be a point on the circle $(x-1)^2+(y-1)^2=1$ such that $(PA)^2+(PB)^2$ have maximum value, then the points, P, A and B lie on:

(a) an ellipse (b) a parabola (c) a straight line (d) a hyperbola

Ans. (c)

Solution:

Let P be $(1+\cos\theta, 1+\sin\theta)$

$$(PA)^2+(PB)^2=(\cos\theta)^2+(\sin\theta-3)^2+(\cos\theta)^2+(\sin\theta+6)^2$$

$$=1-6\sin\theta+9+1+12\sin\theta=36$$

$$=45+6\sin\theta \text{ maximum at } \theta=\frac{\pi}{2}$$

$\therefore$ P(1, 2)

$\therefore$ P, A and B are collinear

12. Let A_1 be the area of the region bounded by the curves y = sin x, y = cos x and y-axis in the first quadrant. Also, let A_2 be the area of the region bounded by the curves y = sin x, y = cos x, x-axis and x = $\frac{\pi}{2}$ in the first quadrant. Then,

(a) $A_1:A_2=1:\sqrt{2}$ and $A_1+A_2=1$ (b) $A_1:A_2=1:2$ and $A_1+A_2=1$

(c) $2A_1=A_2$ and $A_1+A_2=1+\sqrt{2}$ (d) $A_1=A_2$ and $A_1+A_2=\sqrt{2}$

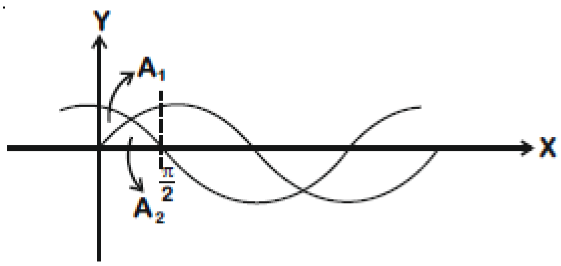
Ans. (a)

Solution:

$$A_1=\int_0^{\frac{\pi}{2}}(\cos x-\sin x)dx=\sqrt{2}-1$$

$$A_2=\int_0^{\frac{\pi}{4}}\sin x\,dx+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\cos x\,dx=\sqrt{2}\left(\sqrt{2}-1\right)$$

$$\therefore A_1:A_2=1:\sqrt{2}\ \&\ A_1+A_2=1$$



13. Let f(x) = $\int_0^x e^t f(t)dt + e^x$ be a differentiable function for all $x \in R$. Then f(x) equals:

(a) $e^{(e^x-1)}$ (b) $2e^{e^x}-1$ (c) $2e^{(e^x-1)}-1$ (d) $e^{e^x}-1$

Ans. (c)

Solution:

Apply Lebnitz' Rule we get

$t^1(x) = ex + (y) + ex$

$$\int \frac{dy}{y+1} = \int e^x dx$$

$$\Rightarrow \ln(y+1) = e^x + c$$

$$\downarrow (0, 1)$$

$$c = \ln\left(\frac{2}{e}\right)$$

$$y+1 = e^{ex}.\frac{2}{e} \Rightarrow y = \left(2.e^{e^x}-1\right)-1$$

14. If 0 < a, b < 1, and $\tan^{-1}a + \tan^{-1}b = \frac{\pi}{4}$, then the value of

$$(a+b)-\left(\frac{a^2+b^2}{2}\right)+\left(\frac{a^3+b^3}{3}\right)-\left(\frac{a^4+b^4}{4}\right)+\ldots \text{ is:}$$

(a) $e^2 - 1$ (b) $\log_e\left(\frac{e}{2}\right)$ (c) e (d) $\log_e 2$

Ans. (d)

Solution:

$$\tan^{-1}a + \tan^{-1}b = \tan^{-1}\left(\frac{a+b}{a-ab}\right) = \frac{\pi}{4}$$

$$\Rightarrow a+b+ab = 1$$

$$\Rightarrow (1+a)(1+b) = 2$$

Given

$$\left(a-\frac{a^2}{2}+\frac{a^3}{3}+\ldots\right)+\left(b-\frac{b^2}{2}+\frac{b^3}{3}+\ldots\right)$$

ln (1 + a) + ln (1 + b)

$$\Rightarrow \ln(1+a)(1+b) = \ln 2$$

15. The sum of the series $\sum_{n=1}^{\infty}\frac{n^2+6n+10}{(2n+1)!}$ is equal to:

(a) $\frac{41}{8}e+\frac{19}{8}e^{-1}-10$ (b) $\frac{41}{8}e-\frac{19}{8}e^{-1}-10$

(c) $-\frac{41}{8}e+\frac{19}{8}e^{-1}-10$ (d) $\frac{41}{8}e+\frac{19}{8}e^{-1}+10$

Ans. (b)

Solution:

$$\sum_{n=1}^{\infty}\frac{n^2+6n+10}{(2n+1)!}=\frac{1}{2}\sum_{n=1}^{\infty}\frac{2n^2+12n+20}{(2n+1)!}$$

$$=\frac{1}{2}\sum_{n=1}^{\infty}\frac{n(2n+1)+11n+20}{(2n+1)!}$$

$$=\frac{1}{2}\sum_{n=1}^{\infty}\frac{n}{(2n)!}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{11n+\frac{11}{2}+\frac{29}{2}}{(2n+1)!}$$

$$=\frac{1}{4}\sum_{n=1}^{\infty}\frac{2n}{(2n)!}+\frac{1}{2}\cdot\frac{11}{2}\sum_{n=1}^{\infty}\frac{2n+1}{(2n+1)!}+\frac{29}{4}\sum_{n=1}^{\infty}\frac{1}{(2n+1)!}$$

$$=\frac{1}{4}\sum_{n=1}^{\infty}\frac{2n}{(2n-1)!}+\frac{11}{4}\sum_{n=1}^{\infty}\frac{1}{2n!}+\frac{29}{4}\sum_{n=1}^{\infty}\frac{1}{(2n+1)!}$$

$$=\frac{1}{4}\left(\frac{e-e^{-1}}{2}\right)+\frac{11}{4}\left(\frac{e+e^{-1}}{2}-1\right)+\frac{29}{4}\left(\frac{e-e^{-1}}{2}-1\right)$$

$$=\frac{15}{2}\left(\frac{e-e^{-1}}{2}\right)+\frac{11}{4}\left(\frac{e+e^{-1}}{2}\right)-10$$

$$=\frac{41}{8}e-\frac{19}{8}e^{-1}-10$$

16. For $x > 0$, if $f(x)=\int_1^x\frac{\log_e t}{(1+t)}dt$, then $f(e)+f\left(\frac{1}{e}\right)$ is equal to:

(a) 1 (b) $\frac{1}{2}$ (c) 0 (d) – 1

Ans. (b)

Solution:

$$f(x)=\int_1^x\frac{\ln t}{1+t}dt$$

then $f\left(\frac{1}{x}\right)=\int_1^{1/x}\frac{\ln t}{1+t}dt$

Let $t=\frac{1}{u}\Rightarrow dt=-\frac{1}{u^2}du$

$$\Rightarrow f\left(\frac{1}{x}\right)=\int_1^x \frac{\ln\frac{1}{u}}{1+\frac{1}{u}}\left(-\frac{1}{u^2}\right)dx$$

$$f\left(\frac{1}{x}\right)=\int_1^x \frac{\ln u}{u(1+u)}du=\int_1^x \frac{\ln t}{t(1+t)}dt$$

$$\therefore\ f(x)+f\left(\frac{1}{x}\right)=\int_1^x \ln t\left(\frac{1}{1+t}+\frac{1}{t(1+t)}\right)dt$$

$$=\int_1^x \ln t\left(\frac{1}{1+t}+\frac{1}{t}-\frac{1}{t+1}\right)dt$$

$$=\int_1^x \frac{\ln t}{t}dt=\frac{1}{2}(\ln x)^2$$

$$\therefore\ f(e)+f\left(\frac{1}{e}\right)=\frac{1}{2}(\ln e)^2=\frac{1}{2}$$

17. If vectors $\vec{a}_1=x\hat{i}-\hat{j}+k$ and $\vec{a}_2=\hat{i}+y\hat{j}+zk$ are collinear, then a possible unit vector parallel to the vector $x\hat{i}+y\hat{j}+zk$ is:

(a) $\frac{1}{\sqrt{3}}\left(\hat{i}-\hat{j}+k\right)$ (b) $\frac{1}{\sqrt{3}}\left(\hat{i}+\hat{j}-k\right)$ (c) $\frac{1}{\sqrt{2}}\left(\hat{i}-\hat{j}\right)$ (d) $\frac{1}{\sqrt{2}}\left(-\hat{j}+k\right)$

Ans. (a)

Solution:

$$\overrightarrow{a_2}=\lambda\overrightarrow{a_1}$$

$$\hat{i}+y\hat{j}+zk=\lambda\left(x\hat{i}-\hat{j}+k\right)$$

$$1=\lambda x,\ y=-\lambda, z=\lambda$$

$$x\hat{i}+y\hat{j}+zk=\frac{1}{\lambda}\hat{i}-\lambda\hat{j}+\lambda k$$

$$\text{Unit vector}=\frac{\frac{1}{\lambda}\hat{i}-\lambda\hat{j}+\lambda k}{\sqrt{\frac{1}{\lambda^2}+\lambda^2+\lambda^2}}$$

$$=\frac{\hat{i}-\lambda^2\hat{j}+\lambda^2 k}{\sqrt{1+2\lambda^4}}$$

Let $\lambda^2=1$, possible unit vector $=\frac{\hat{i}-\hat{j}+k}{\sqrt{3}}$

18. If the locus of the mid-point of the line segment from the point (3, 2) to a point on the circle, $x^2 + y^2 = 1$ is a circle of radius r, then r is equal to:

(a) $\frac{1}{3}$ (b) 1 (c) $\frac{1}{4}$ (d) $\frac{1}{2}$

Ans. (d)

Solution:

Let P(h, k)

Required laws $\frac{3+\cos\theta}{2} = h$ and $\frac{2+\sin\theta}{2} = k$

$\cos\theta = 2h - 3$ and $\sin\theta = 2h - 2$

Squaring and adding we get

$(2h - 3)^2 + (2h - 2)^2 = 1$

$\Rightarrow 4x^2 - 12x + 9 + 4y^2 - 8y + 4 = 1$

$\Rightarrow 4x^2 + 4y^2 - 12x - 8y + 12 = 0$

$\Rightarrow x^2 + y^2 - 3 - 2y + 3 = 0$

Radius = $\sqrt{\frac{9}{4} + 1 - 3} = \frac{1}{2}$

19. Consider the following system of equations:

$x + 2y - 3z = a$

$2x + 6y - 11z = b$

$x - 2y + 7z = c,$

where a, b and c are real constants. Then the system of equations:

(a) has infinite number of solutions when 5a = 2b + c

(b) has a unique solution when 5a = 2b + c

(c) has a unique solution when 5a = 2b + c

(d) has a unique solution for all a, b and c

Ans. (a)

Solution:

$$0 = \begin{vmatrix} 1 & 2 & -3 \\ 2 & 6 & -11 \\ 1 & -2 & 7 \end{vmatrix} = (20) - 2(25) - 3(-10) = 0$$

$x + 2y - 3z = a$ ...(1)

$2x + 6y - 11z = b$ ...(2)

$x - 2y + 7z = c$ ...(3)

5eq (1) = 2eq (2) + eq (3)

it 5a = 2b + c $\Rightarrow$ infinite solution

i.e., it will represent family of planes having a line (of intersection) as a solution

20. Let $f(x) = \sin^{-1}x$ and $g(x) = \dfrac{x^2 - x - 2}{2x^2 - x - 6}$. If $g(2) = \lim_{x\to 2} g(x)$, then the domain of the function fog is:

(a) $(-\infty, -2] \cup \left[-\frac{4}{3}, \infty\right)$ (b) $(-\infty, -2] \cup \left[-\frac{3}{2}, \infty\right)$

(c) $(-\infty, -1] \cup [2, \infty)$ (d) $(-\infty, -2] \cup [-1, \infty)$

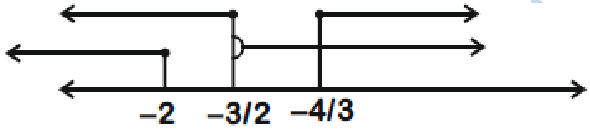
Ans. (a)

Solution:

$$g(2) = \lim_{x\to 2} g(x) = \frac{(x-2)(x+1)}{(2x+3)(x-2)} = \frac{3}{7}$$

$$\log(x) = \sin^{-1}\left(\frac{x+1}{2x+3}\right)$$

for domain $-1 \le \dfrac{x+1}{2x+3} \le 1$



$$\Rightarrow \frac{3x+4}{2x+3} \ge 0 \text{ and } \frac{x+2}{2x+3} \ge 0$$

$x \in (-\infty, -3/2) \cup [-4/3, \infty]$ and $x \in (-\infty, -2] \cup (-3/2, \infty)$

Hence $x \in (-\infty, -2] u (-4/3, \infty]$

Section-II

Numerical Value Type Questions: This section contains 10 questions. In section II, attempt any five questions out of 10. The answer to each question is a NUMERICAL VALUE. For each question, enter the correct numerical value (in decimal notation, truncated/.rounded-off to the second decimal place; e.g. 06.25, 07.00, -00.33, -00.30, 30.27, -27.30) using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.

1. Let α and β be two real numbers such that $\alpha + \beta = 1$ and $\alpha\beta = -1$. Let $p_n = (\alpha)^n + (\beta)^n$, $p_{n-1} = 11$ and pn+1 = 29 for some integer $n \ge 1$. Then, the value of p_n^2 is ________ .

Ans. 324

Solution:

$\because \alpha + \beta = 1$ and $\alpha\beta = -1$

$\therefore$ Equation $x^2 - x = 0$ has two roots α and β.

$\therefore \alpha^2 - \alpha = 1$ and $\beta^2 - \beta = 1$

$\Rightarrow \alpha^{n+1} - \alpha^n = \alpha^{n-1}$ and $\beta^{n+n} - \beta^n = \beta^{n-1}$

$\Rightarrow \alpha^{n+1} + \beta^{n+1} - \alpha^n - \beta^n = \alpha^{n-1} + \beta^{n-1}$

$\Rightarrow P_{n+1} - P_n = P_{n-1}$

$\Rightarrow P_n = 29 - 11$

$\Rightarrow (P_n)^2 = 18^2 = 324$

2. The total number of 4-digit numbers whose greatest common divisor with 18 is 3, is ________.

Ans. 1000

Solution:

Let A denotes a set of number divisible by 3.

B denotes a set of number divisible by 2.

and C denotes a set of number divisible by 9.

Required number of numbers

$= n(A) - n(A \cap B) - n(c) + n(A \cap B \cap C)$

= 3000 – 1500 – 1000 + 500

= 1000

3. Let L be a common tangent line to the curves $4x^2 + 9y^2 = 36$ and $(2x)^2 + (2y)^2 = 31$. Then the square of the slope of the line L is __________ .

Ans. 3

Solution:

Tangent to the curve $\frac{x^2}{9} + \frac{y^2}{14} = 1$ is

$y = mx + \sqrt{9m^2 + 4}$

and equation of tangent to the curve $x^2 + y^2 = \frac{31}{4}$ is

$y = mx + \sqrt{\frac{31}{4}(1 + m^2)}$

for common tangent $9m^2 + 4 = \frac{31}{4} + \frac{31}{4}m^2$

$\Rightarrow \frac{5}{4}m^2 = \frac{15}{4}$

$\Rightarrow m^2 = 3$

4. Let the normals at all the points on a given curve pass through a fixed point (a, b). If the curve passes through (3, –3) and $(4,\ 2\sqrt{2})$, and given that $a - 2\sqrt{2}\ b = 3$, then $(a^2 + b^2 + ab)$ is equal to________ .

Ans. 9

Solution:

Clearly the curve is a circle with centre (a, b)

Centre lies on the line $x - 2\sqrt{2y} = 3$ ...(i)

∵ Circle passes through A(3, – 3) and B(4, $-2\sqrt{2}$)

So centre lies on perpendicular bisector of AB, which is

$x + (3 - 2\sqrt{2})y = 3$ ...(ii)

Clearly x = 3 and y = 0

a = 3 and b = 0

$\Rightarrow a^2 + b^2 + ab = 9$

5. Let $X_1, X_2, ..., X_{18}$ be eighteen observations such that $\sum_{i=1}^{18}(X_i = \alpha) = 36$ and $\sum_{i=1}^{18}(X_i = \beta)^2 = 90$, where α and β are distinct real numbers. If the standard deviation of these observations is 1, then the value of $|\alpha - \beta|$ is____.

Ans. 4

Solution:

$$\because \sum_{i=1}^{18}(x_i - \beta)^2 = 90$$

and $\sum_{i=1}^{18}(x_i - \beta) = \sum_{i=1}^{18}(x_i - \alpha) + 18(\alpha - \beta)$

$= 36 + 18\ (\alpha - \beta)$

So $\text{Var}(x_i) = \text{Var}(x_i - \beta) = \frac{\Sigma(x_i - \beta)^2}{18} - \left(\frac{\Sigma(x_i - \beta)}{18}\right)^2$

$$\Rightarrow 1 = \frac{90}{18} - (2 + \alpha - \beta)^2$$

$$\Rightarrow 2 + \alpha - \beta = \pm 2$$

$$\Rightarrow \alpha - \beta = 0, -4$$

$\because$ α and β are distinct, so $|\alpha - \beta| = 4$

6. If the arithmetic mean and geometric mean of the pth and qth terms of the sequence $-16, 8, -4, 2, \ldots$ satisfy the equation $4x^2 - 9x + 5 = 0$, then p+ q is equal to _________ .

Ans. 10

Solution:

$$T_p = -16\left(-\frac{1}{2}\right)^{p-1} = (-1)^p \,.\, 2^{5-p}$$

and $T_q = (-1)^q \,.\, 2^{5-q}$

$\because$ A.M. of T_p and T_q is $\frac{5}{4}$ and G.M. is 1

$$(-1)^{p+q}\, 2^{10-p-q} = 1 \Rightarrow p + q = 10$$

7. If $I_{m,n}=\int_0^1 x^{m-1}(1-x)^{n-1}dx$, for $m, n \geq 1$, and $\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx=\alpha I_{m,n},\ \alpha \in R$, then α equals ____ .

Ans. 1

Solution:

$\because\ I_{m,n}=\beta_{m,\,n}$

$= \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx$ let $x=\tan^2\theta$

$= \int_0^{\pi/4} \frac{\tan^{2m-2}\theta+\tan^{2n-2}\theta}{\sec^{2(m+n)}\theta}.\ 2\tan\theta\sec^2\theta d\theta$

$= 2\int_0^{\pi/4} \frac{\tan^{2m-1}\theta+\tan^{2n-1}\theta}{\sec^{2(m+n-1)}\theta}\ d\theta$

$= 2\int_0^{\pi/4}\left[\sin^{2m-1}\theta.\cos^{2n-1}\theta+\sin^{2n-1}\theta.\cos^{2m-1}\theta\right]d\theta$

$= 2\int_0^{\pi/2}\sin^{2m-1}\theta.\cos^{2n-1}\theta\ d\theta$

$= \beta_{m,n}$

Clearly $\alpha=1$

8. Let z be those complex numbers which satisfy $|z+5|\leq 4$ and $z(1+i)+\bar{z}(1-i)\geq -10,\ i=\sqrt{-1}$.

If the maximum value of $|z+1|^2$ is $\alpha+\beta\sqrt{2}$, then the value of $(\alpha+\beta)$ is____.

Ans. 48

Solution:

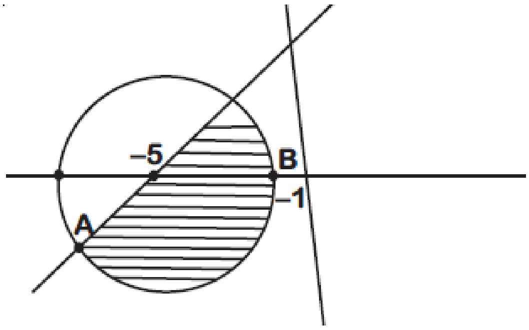
$z(1+i)+\bar{z}\,(1+i)\geq -10\ \Rightarrow\ x-y+5>0$

and $|z+5|\leq 4$ is interior of a circle with centre – 5 and radius 4.

$\because\ |z+1|$ represents the distance of z from – 1.

$|z+1|$ is maximum is z is at A.

z is at A.



$AB^2=|z+1|^2=4^2+4^2-2.4.4.\cos 135^0=32+16\sqrt{2}$

$\Rightarrow\ \alpha=32$ and $\beta=16$

9. If the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 3 & 0 & -1 \end{bmatrix}$ satisfies the equation $A^{20} + \alpha A^{19} + \beta A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ for some real numbers α and β, then $\beta - \alpha$ is equal to _____ .

Ans. 4

Solution:

$\because\ A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A^4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 1 \end{bmatrix}, ...$

So, $A^{20} + \alpha A^{19} + \beta A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^{20} & 0 \\ 0 & 0 & 1 \end{bmatrix} + \alpha \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^{19} & 0 \\ 3 & 0 & -1 \end{bmatrix} + \beta \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 3 & 0 & -1 \end{bmatrix}$

$= \begin{bmatrix} 1+\alpha+\beta & 0 & 0 \\ 0 & 2^{20} + \alpha 2^{19} + 2\beta & 0 \\ 3\alpha + 3\beta & 0 & 1 - \alpha - \beta \end{bmatrix}$

Clearly $\alpha + \beta = 0$ and $2^{20} + \alpha.\, 2^{19} + 2\beta = 4$

$\Rightarrow\ \alpha = -2$ and $\beta = 2$

10. Let a be an integer such that that all the real roots of the polynomial $2x^5 + 5x^4 + 10x^3 + 10x^2 + 10x + 10$ lie in the interval (a, a + 1). Then, |a| is equal to ________ .

Ans. 2

Solution:

Let $f(x) = 2x^5 + 5x^4 + 10(x^3 + x^2 + x + 1)$

$\because\ f(-1) = 3$

and $f(-2) = -34$

hence roots of f(x) lies in (–2, – 1)

Clearly, $|a| = 2$