JEE (Mains) Mathematics Solution

24.2.2021 (Shift-2)

Section-I

Multiple Choice Questions: This section contains 20 multiple choice questions. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.

1. The angle of elevation of a jet plane from a point A on the ground is 60^0. After a flight of 20 seconds at the speed of 432 km/hour, the angle of elevation changes to 30^0. If the jet plane is flying at a constant height, then its height is:

(a) $3600\sqrt{3}$ m (b) $2400\sqrt{3}$ m (c) $1800\sqrt{3}$ m (d) $1200\sqrt{3}$ m

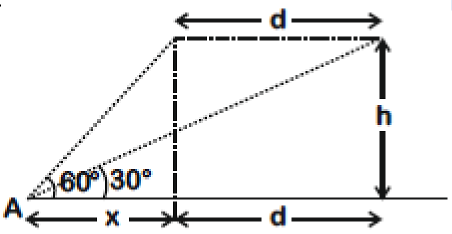
Ans. (d)

Solution:

Given $\tan 30^0 = \frac{h}{x+d}$ and $\tan 60^0 = \frac{h}{x}$

$\Rightarrow x + d = \sqrt{3}h$ and $x = \frac{h}{\sqrt{3}}$

$\Rightarrow d = \left(\sqrt{3} - \frac{1}{\sqrt{3}}\right)h = \frac{2h}{\sqrt{3}}$



Given $\frac{d}{20} = \frac{432 \times 1000}{3600}$

$\Rightarrow d = 2400\text{m}$

$\Rightarrow 2400 = \frac{2h}{\sqrt{3}} \Rightarrow h = 1200\sqrt{3}$ m

2. The probability that two randomly selected subsets of the set {1, 2, 3, 4, 5} have exactly two elements in their intersection, is:

(a) $\frac{65}{2^7}$ (b) $\frac{135}{2^9}$ (c) $\frac{65}{2^8}$ (d) $\frac{35}{2^7}$

Ans. (b)

Solution:

Number of ways of selecting elements common to both A and B = ${}^5C_2$

$\therefore$ Required probability = $\frac{{}^5C_2.3^3}{4^5} = \frac{135}{2^9}$

3. If P is a point on the parabola $y = x^2 + 4$ which is closest to the straight line $y = 4x - 1$, then the co-ordinates of P are:

(a) (−2, 8) (b) (1, 5) (c) (3, 13) (d) (2, 8)

Ans. (d)

Solution:

Closest point will be point on tangency of tangent of same slope i.e. 4

Let equation of tangent y = 4x + c

$\Rightarrow 4x + c = x^2 + 4$ have D = 0

i.e. $x^2 - 4x + (4 - c) = 0$

D = 0 $\Rightarrow 16 - 4(4 - c) = 0 \Rightarrow c = 0$

Tangent is y = 4x gives x = 2 and y = 8 as point of tangency

$\therefore$ Nearest point (2, 8)

4. If $n \geq 2$ is a positive integer, then the sum of the series

$^{n+1}C_2 + 2\left(^2C_2 + {^3C_2} + {^4C_2} + ... + {^nC_2}\right)$ is:

(a) $\frac{n(2n+1)(3n+1)}{6}$ (b) $\frac{n(n-1)(2n+1)}{6}$ (c) $\frac{n(n+1)^2(n+2)}{12}$ (d) $\frac{n(n+1)(2n+1)}{6}$

Ans. (d)

Solution:

Sum of $^2C_2 + {^3C_2} + ... + {^nC_2}$ is coefficient of x^2 in $(1 + x)^2 + (1 + x)^3 + ... + x(1 + x)^n$

i.e. coefficient of x^2 in

$$(1+x)^2 \frac{\left((1+x)^{n-1}-1\right)}{(1+x-1)} = {^{n+1}C_3}$$

Hence required sum = $^{n+1}C_2 + 2.{^{n+1}C_3}$

$= \frac{(n+1)(n)}{2} + \frac{2(n+1)n(n-1)}{6}$

$= \frac{n(n+1)}{2}\frac{(3+2n-2)}{3} = \frac{n(n+1)(2n+1)}{6}$

5. Let f(x) be a differentiable function defined on [0, 2] such that f'(x) = f'(2 – x) for all $x \in (0, 2)$,

f(0) = 1 and f(2) = e^2. Then the value of $\int_0^2 f(x)dx$ is:

(a) $2(1 + e^2)$ (b) $1 + e^2$ (c) $2(1 – e^2)$ (d) $1 – e^2$

Ans. (b)

Solution:

Given f'(x) = f'(2 –x)

f'(x) – f'(2 – x) = 0

Integrating both sides, we get

f(x) + f(2 –x) = c ...(i)

Put x = 0, we get

c = f(0) + f(2) = $1 + e^2$

Integrating 0 to 2 equation (i) both sides, we get

$$\int_0^2 f(x)dx + \int_0^2 f(2-x)dx = \left(1+e^2\right)\times 8 \,\Big|_0^2$$

Also $\int_0^2 f(x)dx = \int_0^2 f(2-x)dx$

Hence $2\int_0^2 f(x)dx = 1+e^2$

$\Rightarrow \int_0^2 f(x)dx = 1+e^2$

6. For the statements p and q, consider the following compound statements:

(a) $(\sim q \wedge (p \to q)) \to \sim p$ (b) $((p \vee q) \wedge \sim p) \to q$

Then which of the following statements is correct?

(a) (a) and (b) both are tautologies. (b) (a) is a tautology but not (b).

(c) (b) is a tautology but not (a). (d) (a) and (b) both are tautologies.

Ans. (a)

Solution:

Truth table for required statements

p	q	$\sim p$	$\sim q$	$p \to q$	$(\sim q) \wedge (p \to q)$	$(\sim q \wedge (p \to q)) \to \sim p$	$p \vee q$	$(p \vee q) \wedge \sim p$	$((p \vee q) \wedge \sim p) \to q$
T	T	F	F	T	F	T	T	F	T
T	F	F	T	F	F	T	T	F	T
F	T	T	F	T	F	T	T	T	T
F	F	T	T	T	T	T (a) is tautology	F	F	T (b) is tautology

7. For the system of linear equations:

x – 2y = 1, x – y + kz = – 2, ky + 4z = 6, $k \in R$, consider the following statements:

(a) The system has unique solution if $k \neq 2, k \neq -2$.

(b) The system has unique solution if k = – 2.

(c) The system has no solution if k = 2.

(d) The system has no solution if k = 2.

(e) The system has finite number of solutions if $k \neq -2$.

Which of the following statements are correct?

(a) (a) and (E) only (b) (a) and (d) only (c) (b) and (e) only (d) (c) and (d) only

Ans. (b)

Solution:

Using Cramer's Rule, we have

$$\Delta = \begin{vmatrix} 1 & -2 & 0 \\ 1 & -1 & k \\ 0 & k & 4 \end{vmatrix} = 1(4-k^2)+2(4) = 4-k^2$$

$$\Delta_x = \begin{vmatrix} 1 & -2 & 0 \\ -2 & -1 & k \\ 6 & k & 4 \end{vmatrix} = 1(4-k^2)+2(-2k+6) = 8-4k-k^2$$

Now, $\Delta = 0$ if $k = \pm 2$

if k = –2, $\Delta = 0$ and $\Delta_x \neq 0$

Hence no solution

Also if k = 2, $\Delta = 0$ and $\Delta_x = 0$

Now

$$\Delta_y = \begin{vmatrix} 1 & 1 & 0 \\ 1 & -2 & k \\ 0 & 6 & 4 \end{vmatrix} = 1(-8-6k)-1(4) = -6k-12 \neq 0$$

Hence, the system has no solution if $k = \pm 2$ and unique solution if $k \neq \pm 2$

8. Let a, b, c be in arithmetic progression. Let the centroid of the triangle with vertices (a, c), (2, b) and (a, b) be $\left(\frac{10}{3}, \frac{7}{3}\right)$. If α, β are the roots of the equation $ax^2 + bx + 1 = 0$, then the value of $\alpha^2+\beta^2-\alpha\beta$ is:

(a) $\frac{69}{256}$ (b) $-\frac{71}{256}$ (c) $-\frac{69}{256}$ (d) $\frac{71}{256}$

Ans. (b)

Solution:

Here, 2b = a + c ...(i)

and centroid of Δ is $\left(\frac{10}{3}, \frac{7}{3}\right)$

$\Rightarrow \frac{a+2+a}{3} = \frac{10}{3} \Rightarrow a = 4$

and $\frac{c+b+b}{3} = \frac{7}{3} \Rightarrow \frac{c+(a+c)}{3} = \frac{7}{3}$ $\Rightarrow 2c + a = 7$

$\Rightarrow 2c + 4 = 7$

$\Rightarrow c = \frac{3}{2}$

So from (i) $2b = \frac{11}{2} \Rightarrow \boxed{b = \frac{11}{4}}$

So the Q.E. if $4x^2 + \frac{11}{4}x + 1 = 0$

$\Rightarrow 16x^2 + 11x + 4 = 0 \quad \Rightarrow \alpha + \beta = \frac{-11}{16}, \alpha\beta = \frac{1}{4}$

Now, $\alpha^2 + \beta^2 - \alpha\beta = \alpha^2 + \beta^2 + 2\alpha\beta - 3\alpha\beta$

$= (\alpha+\beta)^2 - 3\alpha\beta$

$= \left(\frac{-11}{16}\right)^2 - \frac{3}{4}$

$= \frac{121}{256} - \frac{3}{4} = -\frac{71}{256}$

9. A possible value of $\tan\left(\frac{1}{4}\sin^{-1}\frac{\sqrt{63}}{8}\right)$ is:

(a) $2\sqrt{2}-1$ (b) $\sqrt{7}-1$ (c) $\frac{1}{\sqrt{7}}$ (d) $\frac{1}{2\sqrt{2}}$

Ans. (c)

Solution:

$\tan\left(\frac{1}{4}\sin^{-1}\frac{\sqrt{63}}{8}\right) \Rightarrow \tan\left(\frac{\theta}{4}\right) = ?$

Let

$\sin^{-1}\frac{\sqrt{63}}{8} = \theta$ and $\cos\theta = \frac{1}{8} \Rightarrow \sin\theta = \frac{\sqrt{63}}{8} \Rightarrow \tan\theta = \sqrt{63}$ and 6

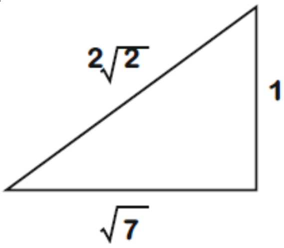
$\Rightarrow 2\cos^2\frac{\theta}{2} - 1 = \frac{1}{8} \Rightarrow 2\cos^2\frac{\theta}{2} = \frac{9}{8} \Rightarrow \cos^2\frac{\theta}{2} = \frac{9}{16}$

$\Rightarrow \cos\frac{\theta}{2} = \frac{3}{4}$

$\Rightarrow 2\cos^2\frac{\theta}{4} - 1 = \frac{3}{4} \qquad \Rightarrow 2\cos^2\frac{\theta}{4} = \frac{7}{4}$

$\Rightarrow \cos^2\frac{\theta}{4} = \frac{7}{8} \Rightarrow \cos\frac{\theta}{4} = \frac{\sqrt{7}}{2\sqrt{2}}$

$\Rightarrow \tan\frac{\theta}{4} = \frac{1}{\sqrt{7}}$



10. Let f: $R \to R$ be defined as

$$f(x) = \begin{cases} -55x, & \text{if } x < -5 \\ 2x^3 - 3x^2 - 120x, & \text{if } -5 \le x \le 4 \\ 2x^3 - 3x^2 - 36x - 336, & \text{if } x > 4, \end{cases}$$

Let A = $\{x \in R :$ f is increasing$\}$. Then A is equal to:

(a) $(-5, \infty)$ (b) $(-\infty, -5) \cup (4, \infty)$

(c) $(-5, -4) \cup (4, \infty)$ (d) $(-\infty, -5) \cup (-4, \infty)$

Ans. (c)

Solution:

$$f(x) = \begin{cases} -55x & , x < -5 \\ 2x^3 - 3x^2 - 120x & , -5 \le x \le 4 \\ 2x^3 - 3x^2 - 336x & , x > 4 \end{cases}$$

Now, f'(x) = $\begin{cases} -55 & , x < -5 \\ 6x^2 - 6x - 120 & , -5 \le x \le 4 \\ 6x^2 - 6x - 36 & , x > 4 \end{cases}$

$$\Rightarrow f'(x) = \begin{cases} -55 & , x < -5 \\ 6(x-5)(x+4) & , -5 < x < 4 \\ 6(x-3)(x+2) & , x > 4 \end{cases}$$

For increasing f'(x) > 0

So clearly f(x) is increasing for $x \in (-5, -4) \cup (4, \infty)$

11. The value of the integral, $\int_1^3 [x^2 - 2x - 2]dx$, where [x] denotes the greatest integer less than or equal to x, is:

(a) – 5 (b) $-\sqrt{2} - \sqrt{3} + 1$

(c) $-\sqrt{2} - \sqrt{3} - 1$ (d) – 4

Ans. (c)

Solution:

$$I = \int_1^3 [x^2 - 2x - 2]dx$$

$$= \int_1^3 \left[\left(x^2 - 2x + 1\right) - 3\right]dx = \int_1^3 \left[(x-1)^2\right]dx - \int_1^3 3dx$$

Now when $x \in [1, 3]$

we see that $(x-1)^2 \in [0, 4]$

So, $I = \int_0^1 0dx + \int_1^{\sqrt{2}} \left[(x-1)^2\right]dx + \int_{\sqrt{2}}^{\sqrt{3}} \left[(x-1)^2\right]dx + \int_{\sqrt{3}}^{2} \left[(x-1)^2\right]dx - \int_1^3 3dx$

$= 0 + \int_1^{\sqrt{2}} 1dx + \int_{\sqrt{2}}^{\sqrt{3}} 2dx + \int_{\sqrt{3}}^{2} 3dx - 6$

$= 0 + [x]_1^{\sqrt{2}} + 2[x]_{\sqrt{2}}^{\sqrt{3}} + 3[x]_{\sqrt{3}}^{2}$

$= \sqrt{2} - 1 + 2\sqrt{3} - 2\sqrt{2} + 6 - 3\sqrt{3} - 6$

$= -\sqrt{3} - \sqrt{2} - 1$

$= -\sqrt{2} - \sqrt{3} - 1$

12. The negation of the statement $\sim p \wedge (p \vee q)$ is:

(a) $p \wedge \sim q$ (b) $\sim p \vee q$ (c) $\sim p \wedge q$ (d) $p \vee \sim q$

Ans. (d)

Solution:

$\sim(\sim p \wedge (p \wedge q)) = p \vee \sim (p \vee q)$

$= p \vee (\sim p \wedge \sim q)$

$= (p \vee \sim p) \wedge (p \vee \sim q)$

$= p \vee \sim q$

13. If the curve $y = ax^2 + bx + c$, $x \in R$, passes through the point (1, 2) and the tangent line to this curve at origin is y = x, then the possible values of a, b, c are:

(a) $a = \frac{1}{2}, b = \frac{1}{2}, c = 1$ (b) a = 1, b = 0, c = 1 (c) a = 1, b = 1, c = 0 (d) a = – 1, b = 1, c = 1

Ans. (c)

Solution:

$y = ax^2 + bx + c$ passes through (1, 2)

So a + b + c = 2 ...(1)

also (0, 0) satisfies $\Rightarrow c = 0$ ...(2)

also slope of tangent at origin is 1 i.e.

$y' = 2ax + b \Rightarrow b = 1$ and $a = 1$

a = 1 = b, c = 0

14. For which of the following curves, the line $x + \sqrt{3}y = 2\sqrt{3}$ is the tangent at the point $\left(\frac{3\sqrt{3}}{2}, \frac{1}{2}\right)$?

(a) $x^2 + 9y^2 = 9$ (b) $y^2 = \frac{1}{6\sqrt{3}}x$ (c) $2x^2 - 18y^2 = 9$ (d) $x^2 + y^2 = 7$

Ans. (a)

Solution:

$x + \sqrt{3}y = 2\sqrt{3}$

$m_t = \frac{-1}{\sqrt{3}}$ and point of tangency $\left(\frac{3\sqrt{3}}{2}, \frac{1}{2}\right)$

Option (1) $x^2+9y^2=9 \Rightarrow \quad 2x+18yy'=0$

$\Rightarrow m_t\left(\frac{3\sqrt{3}}{2},\frac{1}{2}\right)$

i.e. $y'=\frac{-x}{9y}=\frac{-3\frac{\sqrt{3}}{2}}{9\left(\frac{1}{2}\right)}=\frac{-1}{\sqrt{3}}$

Option (2) $y'=\frac{x}{6\sqrt{3}} \quad \Rightarrow \quad y'=\frac{1}{12\sqrt{3}y}$

i.e. $m_t=\frac{1}{6\sqrt{3}}$

Option (3) $2x^2-18y^2=9 \Rightarrow 4x-36yy'=0$

i.e. $m_t=\frac{x}{9y}=\frac{3\sqrt{3}}{2.9\frac{1}{2}}=\frac{1}{\sqrt{3}}$

Option (4) $x^2y^2 = 7 \quad \Rightarrow \quad y'=-\frac{x}{y}$

i.e. $m_t = -3\sqrt{3}$

Hence only option (1) is correct.

15. If a curve y = f(x) passes through the point (1, 2) and satisfies $x\frac{dy}{dx}+y=bx^4$, then for what value of b, $\int_1^2 f(x)dx=\frac{62}{5}$?

(a) $\frac{31}{5}$ (b) $\frac{62}{5}$ (c) 10 (d) 5

Ans. (c)

Solution:

$x\frac{dy}{dx}+y=bx^4$

$\Rightarrow \frac{dy}{dx}+\frac{y}{x}=bx^3$

I.F = $e^{\int\frac{1}{x}dx}=e^{\ln x}=x$

$\Rightarrow yx=\int bx^4dx$

$\Rightarrow xy=\frac{bx^5}{5}+c$

$\downarrow(1,2)$

$\Rightarrow 2=\frac{b}{5}+c \Rightarrow c=2-\frac{b}{5}$

$\therefore y=\frac{bx^4}{5}+\frac{1}{x}\left(2-\frac{b}{5}\right)$

$$\int_1^2 f(x)dx = \frac{bx^5}{25}\Big|_1^2 + \left(2-\frac{b}{5}\right)\ln x\Big|_1^2$$

$$\frac{31b}{25} + \left(2-\frac{b}{5}\right)\ln 2 = \frac{62}{5}$$

$$\Rightarrow \left(2-\frac{b}{5}\right)\ln 2 = \left(2-\frac{b}{5}\right)\frac{31}{5}$$

$$\Rightarrow 2-\frac{b}{5} = 0 \Rightarrow b = 10$$

16. The area of the region: $R = \{(x, y): 5x^2 \le y \le 2x^2 + 9\}$ is:

(a) $6\sqrt{3}$ square units (b) $11\sqrt{3}$ square units (c) $12\sqrt{3}$ square units (d) $9\sqrt{3}$ square units

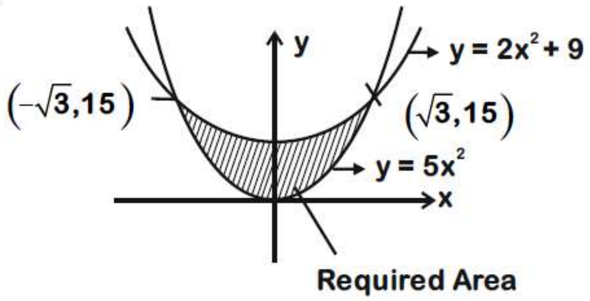
Ans. (c)

Solution:

Required Area = $2\int_0^{\sqrt{3}}\left(\left(2x^2+9\right)-\left(5x^2\right)\right)dx$

$= 2\left(9x - \frac{3x^3}{3}\right)_0^{\sqrt{3}}$

$= 2\left(9\sqrt{3}-3\sqrt{3}\right) = 12\sqrt{3}$ sq. units



17. The vector equation of the plane passing through the intersection of the planes $\vec{r}.\left(\hat{i}+\hat{j}+k\right)=1$ and $\vec{r}.\left(\hat{i}-2\hat{j}\right)=-2$, and the point (1, 0, 2) is:

(a) $\vec{r}.\left(\hat{i}+7\hat{j}+3k\right)=\frac{7}{3}$ (b) $\vec{r}.\left(3\hat{i}+7\hat{j}+3\hat{k}\right)=7$ (c) $\vec{r}.\left(\hat{i}-7\hat{j}+3k\right)=\frac{7}{3}$ (d) $\vec{r}.\left(\hat{i}+7\hat{j}+3k\right)=7$

Ans. (d)

Solution:

$P_1 + \lambda P_2 = 0$

$\Rightarrow \left(\vec{r}.\left(\hat{i}+\hat{j}+k\right)-1\right)+\lambda\left(\vec{r}.\left(\hat{i}-2\hat{j}\right)+2\right)=0$ ...(1)

$\downarrow \left(\hat{i}+2k\right)$

$\Rightarrow (1+2-1)+\lambda(1+2)=0 \Rightarrow \lambda = -\frac{2}{3}$ ...(2)

by (1) and (2)

$\vec{r}.\left(\hat{i}+\hat{j}+k-\frac{2}{3}\left(\hat{i}-2\hat{j}\right)\right)-1-\frac{4}{3}=0$

$\Rightarrow \vec{r}.\left(\frac{\hat{i}}{3}+\frac{7}{3}\hat{j}+k\right)=\frac{7}{3}$

$\Rightarrow \vec{r}.\left(\hat{i}+7\hat{j}+3k\right)=7$

18. Let A and B be 3×3 real matrices such that A is symmetric matrix and B is skew-symmetric matrix. Then the system of linear equations $(A^2B^2 - B^2A^2)X = O$, where X is a 3×1 column matrix of unknown variables and O is a 3×1 null matrix, has:

(a) exactly two solutions (b) infinitely many solutions

(c) no solution (d) a unique solution

Ans. (b)

Solution:

Let $C = A^2B^2 - B^2A^2$

Then $C^T = (A^2B^2 - B^2A^2)^T$

$= (B^T)^2.(A^T)^2 - (A^T)^2.(B^T)^2$

$= (-B)^2 A^2 - A^2.(-B)^2 \quad \{\because A^T = A \text{ and } B^T = -B\}$

$= (B)^2 A^2 - A^2B^2$

$\therefore C + C^T = 0$

$\therefore$ C is a skew symmetric odd order matrix

$\therefore |C| = |A^2B^2 - B^2A^2| = 0$

$\therefore$ Equation $(A^2B^2 - B^2A^2)X = 0$ has

Infinite many solution

19. Let $a, b\in R$. If the mirror image of the point P(a, 6, 9) with respect to the line $\frac{x-3}{7}=\frac{y-2}{5}=\frac{z-1}{-9}$ is $(20, b, -a, -9)$, then $|a+b|$ is equal to:

(a) 86 (b) 88 (c) 90 (d) 84

Ans. (b)

Solution:

$\because$ Mirror image of P(a, 6, 9) is Q(20, b, –a – 9).

So mid-point of PQ $\left(\text{i.e,}\left(\frac{a}{2}+10, \frac{b}{2}+3, -\frac{a}{2}\right)\right)$ lies on the given line

$$\frac{\frac{a}{2}+7}{7}=\frac{\frac{b}{2}+1}{5}=\frac{-\frac{a}{2}-1}{-9} \Rightarrow a=-56 \text{ and } b=-32$$

$a + b = -88$

$|a+b| = 88$

20. Let f be a twice differentiable function defined on R such that f(0) = 1, f ′(0) = 2 and f′(x) $\neq 0$ for all $x\in R$. If $\begin{vmatrix} f(x) & f'(x) \\ f'(x) & f''(x) \end{vmatrix}=0$, for all $x\in R$, then the value of f(1) lies in the interval:

(a) (0, 3) (b) (9, 12) (c) (3, 6) (d) (6, 9)

Ans. (d)

Solution:

$$\because \begin{vmatrix} f(x) & f'(x) \\ f'(x) & f''(x) \end{vmatrix}=0,$$

$$\Rightarrow \frac{f(x).f''(x)-(f'(x))^2}{(f(x))^2}=0 \quad \because f(x)\neq 0.$$

$\therefore \frac{d}{dx}\left(\frac{f'(x)}{f(x)}\right)=0.$

$\Rightarrow \frac{f'(x)}{f(x)}=c,$ $\because f(0)=1$ and $f'(0)=2$

$\therefore c=2$

$\therefore f'(x) = 2f(x)$

$\Rightarrow \frac{f'(x)}{f(x)}=2$

$\Rightarrow \ln|f(x)|=2x+d.$

$\Rightarrow |f(x)|=e^{2x}.$ $\because f(0)=1.$

$\therefore f(1)=e^2$

$\therefore f(1)\in(6,9)$

Section-II

Numerical Value Type Questions: This section contains 10 questions. In section II, attempt any five questions out of 10. The answer to each question is a NUMERICAL VALUE. For each question, enter the correct numerical value (in decimal notation, truncated/.rounded-off to the second decimal place; e.g. 06.25, 07.00, -00.33, -00.30, 30.27, -27.30) using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.

1. The sum of first four terms of a geometric progression (G.P) is $\frac{65}{12}$ and the sum of their respectively reciprocals is $\frac{65}{18}$. If the product of first three terms of the G.P. is 1, and the third term is α, then 2α is____ .

Ans. 3

Solution:

Let the G.P. be ar^3, ar, $\frac{a}{r}, \frac{a}{r^3}$,....

$\because a\left(r^3+r+\frac{1}{r}+\frac{1}{r^3}\right)=\frac{65}{12}$ ...(i)

$\frac{1}{a}\left(r^3+r+\frac{1}{r}+\frac{1}{r^3}\right)=\frac{65}{18}$ ...(ii)

$\Rightarrow a^2=\frac{3}{2}$

Also $a^3r^3=1 \Rightarrow r=\frac{1}{a}$

and $\alpha=\frac{a}{r}=a^2=\frac{3}{2}$

$\Rightarrow 2\alpha=3$

2. Let λ be an integer. If the shortest distance between the lines $x-\lambda=2y-1=-2z$ and $x=y+2\lambda=z-\lambda$ is $\frac{\sqrt{7}}{2\sqrt{2}}$, then the value of $|\lambda|$ is ________ .

Ans. 1

Solution:

$$L_1: \frac{x-\lambda}{1}=\frac{y-\frac{1}{2}}{\frac{1}{2}}=\frac{z}{-\frac{1}{2}}$$

$$L_2: \frac{x}{1}=\frac{y-2\lambda}{1}=\frac{z-\lambda}{1}$$

$$\overline{b_1}\times\overline{b_2}=\begin{vmatrix}\hat{i} & \hat{j} & k\\ 1 & 1/2 & -1/2\\ 1 & 1 & 1\end{vmatrix}=\hat{i}-\frac{3}{2}\hat{j}+\frac{1}{2}k$$

$$\overline{a_1}-\overline{a_2}=\lambda\hat{i}+\left(\frac{1}{2}+2\lambda\right)\hat{j}-\lambda k$$

$$d=\left|\frac{(\overline{a_1}-\overline{a_2}).(\overline{b_1}-\overline{b_2})}{(\overline{b_1}\times\overline{b_2})}\right|=\left|\frac{\frac{5\lambda}{2}+\frac{3}{4}}{\sqrt{\frac{7}{2}}}\right|=\frac{\sqrt{7}}{2\sqrt{2}}$$

$$\Rightarrow \left|\frac{5}{4}\lambda+\frac{3}{4}\right|=\frac{7}{4}\Rightarrow\lambda=\frac{2}{5}\text{ or }-1$$

$$\Rightarrow|\lambda|=1$$

3. Let $i=\sqrt{-1}$. If $\frac{(-1+i\sqrt{3})^{21}}{(1-i)^{24}}+\frac{(1+i\sqrt{3})^{21}}{(1+i)^{24}}=k$, and $n=[|k|]$ be the greatest integral part of $|k|$.

Then $\sum_{j=0}^{n+5}(j+5)^2-\sum_{j=0}^{n+5}(j+5)$ is equal to ________ .

Ans. 310

Solution:

$$k=\frac{(-1+i\sqrt{3})^{21}}{(1-i)^{24}}+\frac{(1+i\sqrt{3})^{21}}{(1+i)^{24}}$$

$$=\frac{\left(2e^{\frac{2\pi}{3}i}\right)^{21}}{\left(\sqrt{2}e^{-\frac{\pi}{4}i}\right)^{24}}+\frac{\left(2e^{\frac{\pi}{3}i}\right)^{21}}{\left(\sqrt{2}e^{\frac{\pi}{4}i}\right)^{24}}$$

$$=2^9\left[e^{(14\pi+6\pi)i}+e^{(7\pi-6\pi)i}\right]$$

$$=2^9[0]=0$$

Now $2\sum_{j=0}^{5}\frac{1}{3}\{(j+4)(j+5)(j+6)-(j+3)(j+4)(j+5)\}$

$$=\frac{1}{3}[9\times10\times11-3\times4\times5]=310$$

4. The number of the real roots of the equation $(x+1)^2+|x-5|=\frac{27}{4}$ is_____.

Ans. 2

Solution:

If $x \geq 5$

$(x^2+2x+1)+(x-5)=\frac{27}{4}$

$\Rightarrow\ x^2+3x-\frac{43}{4}=0$

$\Rightarrow\ x=\frac{-3\pm\sqrt{52}}{2}$ (No value is greater than or equal to 5)

If $x < 5$

$x^2+2x+1-x+5=\frac{27}{4}$

$\Rightarrow\ x^2+x-\frac{3}{4}=0$

$\Rightarrow\ x=\frac{1}{2},-\frac{3}{2}$

So there will be two real roots.

5. The students S_1, S_2, ….,S_{10} are to be divided into 3 groups A, B and C such that each group has at least one student and the group C has at most 3 students. Then the total number of possibilities of forming such groups is _________ .

Ans. 31650

Solution:

Number of possible ways when

(i) There is one student in group C = ${}^{10}C_1.\ (2^9-2)=5100$

(ii) There are two students in group C = ${}^{10}C_2.\ (2^8-2)=11430$

(iii) There are three students in group C =${}^{10}C_3.\ (2^7-2)=15120$

Total number of ways = 31650

6. Let a point P be such that its distance from the point (5, 0) is thrice the distance of P from the point (–5, 0). If the locus of the point P is a circle of radius r, then $4r^2$ is equal to ________ .

Ans. 56.25

Solution:

Let P(x, y)

$\sqrt{(x-5)^2+y^2}=3\sqrt{(x+5)^2+y^2}$

$\Rightarrow\ x^2+25-10x+y^2=9(x^2+y^2+10x+25)$

$\Rightarrow 8x^2+8y^2+100x+200=0$

$\Rightarrow\ x^2+y^2+\frac{25}{2}x+25=0$

$r=\sqrt{\left(\frac{25}{4}\right)^2-25}=5\left(\frac{3}{4}\right)=\frac{15}{4}$

$\Rightarrow 4r^2=56.25$

7. For integers n and r, let

$$\binom{n}{r} = \begin{cases} {}^nC_r, \text{ if } n \ge r \ge 0 \\ 0, \quad \text{otherwise} \end{cases}$$

The maximum value of k for which the sum

$$\sum_{i=0}^{k}\binom{10}{i}\binom{15}{k-i} + \sum_{i=0}^{k+1}\binom{12}{i}\binom{13}{k+i-i}$$ exists, is equal to ________.

Ans. 12

Solution:

$$\sum_{i=0}^{k} {}^{10}C_1 . {}^{15}C_{k-1} + \sum_{i=0}^{k+1} {}^{12}C_1 . {}^{13}C_{k+1-i}$$

$= {}^{25}C_k + {}^{25}C_{k+1}$

$= {}^{26}C_{k+1}$

$0 \le k+1 \le 25$

$-1 \le k \le 24$

But ${}^{13}C_{k+1-i}$ exists for $0 \le i \le k+1$

then $0 \le i \le k+1$

$\Rightarrow k \le 12$

Hence $k_{max} = 12$

8. If $a+\alpha = 1, b+\beta = 2$ and $af(x)+\alpha f\left(\frac{1}{x}\right) = bx + \frac{\beta}{x}$, $x \ne 0$, then the value of the expression

$$\frac{f(x)+f\left(\frac{1}{x}\right)}{x+\frac{1}{x}}$$ is______.

Ans. 2

Solution:

$$af(x)+\alpha f\left(\frac{1}{x}\right) = bx + \frac{\beta}{x} \qquad \text{...(i)}$$

Replace x by $\frac{1}{x}$

$$af\left(\frac{1}{x}\right) + \alpha f(x) = \frac{b}{x} + \beta x \qquad \text{...(ii)}$$

(i) + (ii)

$$\Rightarrow (a+\alpha)\left(f(x)+f\left(\frac{1}{x}\right)\right) = (b+\beta)\left(x+\frac{1}{x}\right)$$

$$\Rightarrow \frac{f(x)+f\left(\frac{1}{x}\right)}{x+\frac{1}{x}} = \frac{b+\beta}{a+\alpha} = \frac{2}{1} = 2$$

9. If the area of the triangle formed by the positive x-axis, the normal and the tangent to the circle $(x - 2)^2 + (y - 3)^2 = 25$ at the point (5, 7) is A, then 24A is equal to ________ .

Ans. 1225

Solution:

Equation of normal PN

$$Y - 7 = \frac{7-3}{5-2}(x-5)$$

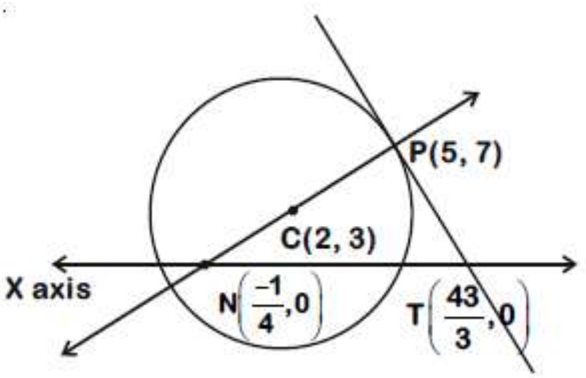
$4x - 3y + 1 = 0$ ...(i)

$$N\left(\frac{-1}{4}, 0\right)$$

Equation of Tangent PT

$3x + 4y = 43$ ...(ii)

$$T\left(\frac{43}{3}, 0\right)$$



$$PT = \frac{43}{3} + \frac{1}{4} = \frac{175}{12}$$

Area of triangle PNT = $\frac{1}{2} \times \frac{175}{12} \times 7 = A$

$24A = 1225$

10. If the variance of 10 natural numbers 1, 1, 1, ..., 1, k is less than 10, then the maximum possible value of k is________ .

Ans. 11

Solution:

$$\sigma^2 = \frac{9+k^2}{10} - \left(\frac{9+k^2}{10}\right)^2 < 10$$

$10(k^2 + 9) - (k^2 + 18k + 81) < 1000$

$9k^2 - 18k + 9 < 1000$

$9(k - 1)^2 < 1000$

$$|k-1| < \frac{10\sqrt{10}}{3} = \frac{10 \times 3.162}{3} = 10.54$$

$-10.54 < k - 1 < 10.54$

$-9.54 < k < 11.54$

But $k \in N$, $\therefore\ k_{max} = 11$