JEE-Main Mathematics Solution 17.3.2021 Shift-2

1. Let f: $R \to R$ be defined as $f(x) = e^{-x}\sin x$. If F: $[0, 1] \to R$ is a differentiable function such that $F(x) = \int_0^x f(t)dt$, then the value of $\int_0^1 \left(F'(x)+f(x)\right)e^x dx$ lies in the interval

(a) $\left[\frac{327}{360}, \frac{329}{360}\right]$ (b) $\left[\frac{330}{360}, \frac{331}{360}\right]$ (c) $\left[\frac{331}{360}, \frac{334}{360}\right]$ (d) $\left[\frac{335}{360}, \frac{336}{360}\right]$

Ans. (b)

Solution:

$f(x) = e^{-x}\sin x$

Now, $F(x) = \int_0^x f(t)dt \quad \Rightarrow F'(x) = f(x)$

$I = \int_0^1 \left(F'(x)+f(x)\right)e^x\,dx = \int_0^1 \left(f(x)+f(x)\right).e^x\,dx$

$= 2\int_0^1 f(x).\,e^x\,dx = 2\int_0^1 e^{-x}\sin x.\,e^x\,dx$

$= 2\int_0^1 \sin x\,dx$

= 2(1 – cos 1)

$I = 2\left\{1-\left(1-\frac{1}{\lfloor 2}+\frac{1}{\lfloor 4}+\frac{1}{\lfloor 6}+\frac{1}{\lfloor 8}\ldots\ldots\right)\right\}$

$I = 1-\frac{2}{\lfloor 4}+\frac{2}{\lfloor 6}-\frac{2}{\lfloor 9}+\ldots.$

$1-\frac{2}{\lfloor 4} < I < 1-\frac{2}{\lfloor 4}+\frac{2}{\lfloor 6}$

$\frac{11}{12} < I < \frac{331}{360}$

$\Rightarrow I \in \left[\frac{11}{12}, \frac{331}{360}\right]$

$\Rightarrow I \in \left[\frac{330}{360}, \frac{331}{360}\right]$

2. If the integral $\int_0^{10} \frac{[\sin 2\pi x]}{e^{x-[x]}} dx = \alpha e^{-1} + \beta e^{-\frac{1}{2}} + \gamma$, where α, β, γ are integers and [x] denotes the greatest integer less than or equal to x, then the value of $\alpha+\beta+\gamma$ is equal to:

(a) 0 (b) 20 (c) 25 (d) 10

Ans. (a)

Solution:

Let I = $\int_0^{10} \frac{[\sin 2\pi x]}{e^{x-[x]}} dx = \int_0^{10} \frac{[\sin 2\pi x]}{e^{\{x\}}} dx$

Function f(x) = $\frac{[\sin 2\pi x]}{e^{\{x\}}}$ is periodic with period '1'

Therefore

$I = 10\int_0^1 \frac{[\sin 2\pi x]}{e^{\{x\}}} dx$

$= 10\int_0^1 \frac{[\sin 2\pi x]}{e^{x}} dx$

$= 10\left(\int_0^{1/2} \frac{[\sin 2\pi x]}{e^{x}} dx + \int_{1/2}^{1} \frac{[\sin 2\pi x]}{e^{x}} dx\right)$

$= 10\left(0 + \int_{1/2}^{1} \frac{(-1)}{e^{x}} dx\right)$

$= -10\int_{1/2}^{1} e^{-x} dx$

$= 10\left(e^{-1} - e^{-1/2}\right)$

Now,

$10.e^{-1} - 10.\,e^{-1/2} = \alpha e^{-1} + \beta e^{-1/2} + \gamma$ (given)

$\Rightarrow \alpha = 10, \beta = -10, \gamma = 0$

$\Rightarrow \alpha + \beta + \gamma = 0$

3. Let y = y(x) be the solution of the differential equation cos x (3 sin x + cos x + 3) dy = (1 + y sin x (3 sin x + cos x + 3) dx, $0 \le x \le \frac{\pi}{2}, y(0) = 0$. Then, $y\left(\frac{\pi}{3}\right)$ is equal to:

(a) $2\log_e\left(\frac{2\sqrt{3}+9}{6}\right)$ (b) $2\log_e\left(\frac{2\sqrt{3}+10}{11}\right)$ (c) $2\log_e\left(\frac{\sqrt{3}+7}{2}\right)$ (d) $2\log_e\left(\frac{3\sqrt{3}-8}{4}\right)$

Ans. (b)

Solution:

cos x(3 sin x + cos x + 3) dy

= (1+ y sin x(3sin x + cos x + 3))dx

$\frac{dy}{dx} - (\tan x)y = \frac{1}{(3\sin x + \cos x + 3)\cos x}$

I.F. = $e^{\int -\tan x\, dx} = e^{\ell n |\cos x|} = |\cos x|$

$= \cos x \ \forall\ x \in \left[0, \frac{\pi}{2}\right)$

Solution of D.E.

$$y(\cos x) = \int (\cos x).\frac{1}{\cos x(3\sin x + \cos x + 3)}\, dx + C$$

$$y(\cos x) = \int \frac{dx}{3\sin x + \cos x + 3}\, dx + C$$

$$y(\cos x) = \int \frac{\left(\sec^2 \frac{x}{2}\right)}{2\tan^2 \frac{x}{2} + 6\tan \frac{x}{2} + 4}\, dx + C$$

Now

$$\text{Let } I_1 = \int \frac{\left(\sec^2 \frac{x}{2}\right)}{2\left(\tan^2 \frac{x}{2} + 3\tan \frac{x}{2} + 2\right)}\, dx + C$$

Put $\tan \frac{x}{2} = t \ \Rightarrow \frac{1}{2}\sec^2 \frac{x}{2}\, dx = dt$

$$I_1 = \int \frac{dt}{t^3 + 3t + 2} = \int \frac{dt}{(t+2)(t+1)}$$

$$= \int \left(\frac{1}{t+1} - \frac{1}{t+2}\right) dt$$

$$= \ell n \left|\left(\frac{t+1}{t+2}\right)\right| = \ell n \left|\left(\frac{\tan \frac{x}{2} + 1}{\tan \frac{x}{2} + 2}\right)\right|$$

So solution of D. E.

$$y(\cos x) = \ell n \left|\frac{1 + \tan \frac{x}{2}}{2 + \tan \frac{x}{2}}\right| + C$$

$$\Rightarrow y(\cos x) = \ell n \left(\frac{1 + \tan \frac{x}{2}}{2 + \tan \frac{x}{2}}\right) + C \text{ for } 0 \le x < \frac{\pi}{2}$$

Now, it is given y(0) = 0

$$\Rightarrow 0 = \ell n \left(\frac{1}{n}\right) + C \qquad \Rightarrow \qquad \boxed{C = \ell n 2}$$

$$\Rightarrow y(\cos x) = \ell n\left(\frac{1+\tan\frac{x}{2}}{2+\tan\frac{x}{2}}\right) + \ell n2$$

For $x = \frac{\pi}{3}$

$$y\left(\frac{1}{2}\right) = \ell n\left(\frac{1+\frac{1}{\sqrt{3}}}{2+\frac{1}{\sqrt{3}}}\right) + \ell n2$$

$$y = 2\ell n\left(\frac{2\sqrt{3}+10}{11}\right)$$

4. The value of $\sum_{r=0}^{6}\left({}^6C_r \cdot {}^6C_{6-r}\right)$ is equal to:

(a) 1124 (b) 1324 (c) 1024 (d) 924

Ans. (d)

Solution:

$\sum_{r=0}^{6} {}^6C_r \cdot {}^6C_{6-r}$

$= {}^6C_0 \cdot {}^6C_6 + {}^6C_1 \cdot {}^6C_5 + \ldots\ldots + {}^6C_6 \cdot {}^6C_0$

Now,

$(1 + x)^6 (1 + x)^6$

$= ({}^0C_0 + {}^6C_1x + {}^6C_2x^2 + \ldots\ldots + {}^6C_6x^6)$

$({}^6C_0 + {}^6C_1x + {}^6C_2x^2 + \ldots . + {}^6C_6x^6)$

Comparing coefficient of x^6 both sides

${}^6C_0 \cdot {}^6C_6 + {}^6C_1 + {}^6C_5 + \ldots\ldots + {}^6C_6 \cdot {}^6C_0 = {}^{12}C_6$

= 924.

5. The value of $\lim_{n\to\infty}\frac{[r]+[2r]+\ldots\ldots+[nr]}{n^2}$, where r is non-zero real number and [r] denotes the greatest integer less than or equal to r, is equal to:

(a) $\frac{r}{2}$ (b) r (c) 2r (d) 0

Ans. (a)

Solution:

We know that

$r \le [r] < r+1$

and $2r \le [2r] < 2r+1$

$3r \le [3r] < 3r+1$

$\vdots \quad \vdots \quad \vdots$

$nr \le [nr] < nr+1$

$r+2r+...+nr$

$\le [r]+[2r]+....+[nr]<(r+2r+....+nr)+n$

$$\frac{\frac{n(n+1)}{2}}{n^2} \le \frac{[r]+[2r]+.....+[nr]}{n^2} < \frac{\frac{n(n+1)}{2}r+n}{n^2}$$

Now,

$$\lim_{n\to\infty} \frac{n(n+1).r}{2.n^2} = \frac{r}{2}$$

and $$\lim_{n\to\infty} \frac{\frac{n(n+1)r}{2}+n}{n^2} = \frac{r}{2}$$

So, by Sandwich Theorem, we can conclude that

$$\lim_{n\to\infty} \frac{[r]+[2r]+....+[nr]}{n^2} = \frac{r}{2}$$

6. The number of solutions of the equation $\sin^{-1}\left[x^2+\frac{1}{3}\right]+\cos^{-1}\left[x^2-\frac{2}{3}\right]=x^2$, for $x\in[-1, 1]$, and $[x]$ denotes the greatest integer less than or equal to x, is:

(a) 2 (b) 0 (c) 4 (d) Infinite

Ans. (b)

Solution:

Given equation

$$\sin^{-1}\left[x^2+\frac{1}{3}\right]+\cos^{-1}\left[x^2-\frac{2}{3}\right]=x^2$$

Now, $\sin^{-1}\left[x^2+\frac{1}{3}\right]$ is defined if

$$-1\le x^2+\frac{1}{3}<2 \Rightarrow \frac{-4}{3}\le x^2<\frac{5}{3}$$

$$\Rightarrow \boxed{0\le x^2<\frac{5}{3}} \quad ....(1)$$

and $\cos^{-1}\left[x^2-\frac{2}{3}\right]$ is defined if

$$-1\le x^2-\frac{2}{3}<2 \Rightarrow \frac{-1}{3}\le x^2<\frac{8}{3}$$

$$\Rightarrow \boxed{0\le x^2<\frac{8}{3}} \quad ....(2)$$

So, form (1) and (2) we can conclude $\boxed{0\le x^2<\frac{5}{3}}$

Case-I if $0\le x^2<\frac{2}{3}$

$$\sin^{-1}(0)+\cos^{-1}(-1)=x^2$$

$\Rightarrow x + \pi = x^2$

$\Rightarrow x^2 = \pi$

but $\pi \notin \left[0, \frac{2}{3}\right)$

$\Rightarrow$ No value of 'x'

Case-II If $\frac{2}{3} \le x^2 < \frac{5}{3}$

$\sin^{-1}(1) + \cos^{-1}(0) = x^2$

$\Rightarrow \frac{\pi}{2} + \frac{\pi}{2} = x^2$

$\Rightarrow x^2 = \pi$

but $\pi \notin \left[\frac{2}{3}, \frac{5}{3}\right)$

$\Rightarrow$No value of 'x'

So, number of solutions of the equation is zero.

7. Let a computer program generate only the digits 0 and 1 to form a string of binary numbers with probability of occurrence of 0 at even places be $\frac{1}{2}$ and probability of occurrence of 0 at the odd place be $\frac{1}{3}$. Then the probability that '10' is followed by '01' is equal to:

(a) $\frac{1}{18}$ (b) $\frac{1}{3}$ (c) $\frac{1}{6}$ (d) $\frac{1}{9}$

Ans. (d)

Solution:

$\underset{\text{odd place}}{1} \quad \underset{\text{even place}}{0} \quad \underset{\text{odd place}}{0} \quad \underset{\text{even place}}{1}$

or $\underset{\text{even place}}{1} \quad \underset{\text{odd place}}{0} \quad \underset{\text{even place}}{0} \quad \underset{\text{odd place}}{1}$

$\Rightarrow \left(\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{2}{3}\right) + \left(\frac{2}{2} \cdot \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{2}\right)$

$\Rightarrow \frac{1}{9}$

8. The number of solutions of the equation x + 2 tan x = $\frac{\pi}{2}$ in the interval $[0, 2\pi]$ is:

(a) 3 (b) 4 (c) 2 (d) 5

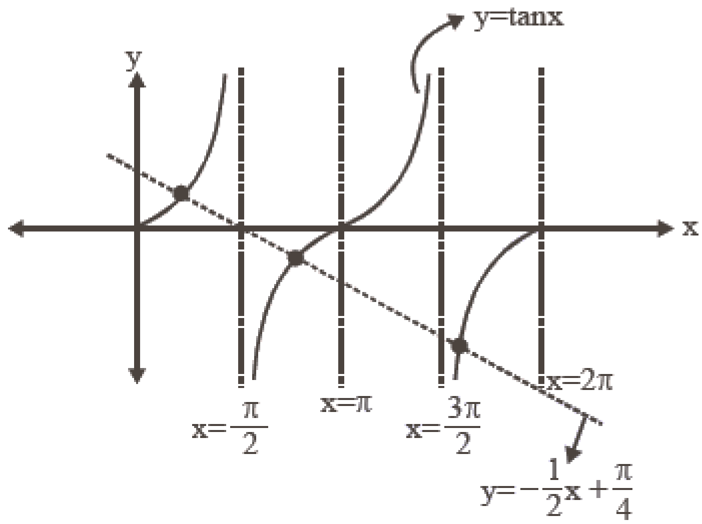
Ans. (a)

Solution:

$x + 2\tan x = \frac{\pi}{2}$

$\Rightarrow 2\tan x = \frac{\pi}{2} - x$

$\Rightarrow \tan x = -\frac{1}{2}x + \frac{\pi}{4}$



Number of solutions of the given equation is '3'.

9. Let S_1, S_2 and S_3 be three sets defined as

$S_1 = \{z \in C : |z-1| \le \sqrt{2}\}$

$S_2 = \{z \in C : \text{Re}((1-i)z) \ge 1\}$

$S_3 = \{z \in C : \text{Im}(z) \le 1\}$

Then the set $S_1 \cap S_2 \cap S_3$

(a) is a singleton

(b) has exactly two elements

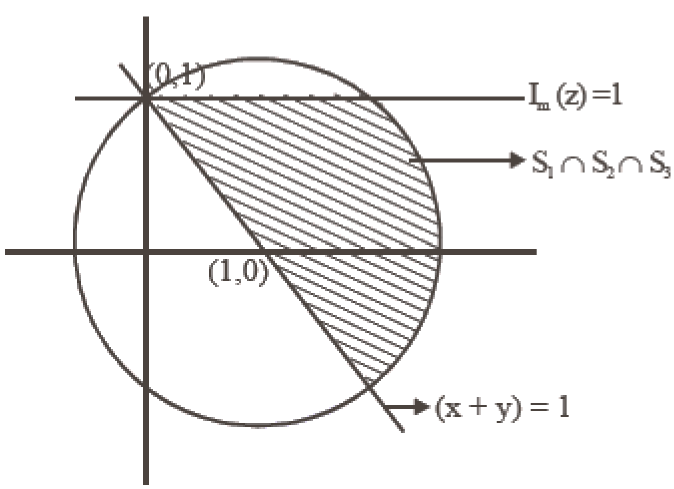
(c) has infinitely many elements

(d) has exactly three elements

Ans. (c)

Solution:

For $|z-1| \le \sqrt{2}, z$ lies on and inside the circle of radius $\sqrt{2}$ units and centre (1, 0).



For S_2

Let z = x + iy

Now, (1 – i) (z) = (1 – i) (x + iy)

Re(1 – i)z) = x + y

$\Rightarrow x+y\geq 1$

$\Rightarrow S_1 \cap S_2 \cap S_3$ has infinity many elements

10. If the curve y = y(x) is the solution of the differential equation

$2(x^2 + x^{5/4})dy - y(x + x^{1/4})(dx = 2x^{9/4}$, x > 0 which passes through the point $\left(1, 1-\frac{4}{3}\log_e 2\right)$,

then the value of y(16) is equal to:

(a) $4\left(\frac{31}{3}+\frac{8}{3}\log_e 3\right)$ (b) $\left(\frac{31}{3}+\frac{8}{3}\log_e 3\right)$ (c) $4\left(\frac{31}{3}-\frac{8}{3}\log_e 3\right)$ (d) $\left(\frac{31}{3}-\frac{8}{3}\log_e 3\right)$

Ans. (c)

Solution:

$$\frac{dy}{dx}-\frac{y}{2x}=\frac{x^{9/4}}{x^{5/4}(x^{3/4}+1)}$$

$$\text{IF} = e^{-\int\frac{dx}{2d}} = e^{-\frac{1}{2}\ln x} = \frac{1}{x^{1/2}}$$

$$y.\,x^{-1/2}=\int\frac{x^{9/4}.\,x^{-1/2}}{x^{5/4}\left(x^{3/4}+1\right)}dx$$

$$\int\frac{x^{1/2}}{\left(x^{3/4}+1\right)}\,dx$$

$$x=t^4 \Rightarrow dx = 4t^3\ dt$$

$$\int\frac{t^2.\,4t^3\ dt}{\left(t^3+1\right)}$$

$$4\int\frac{t^2\left(t^3+1-1\right)}{\left(t^3+1\right)}dt$$

$$4\int t^2dt-4\int\frac{t^2}{t^3+1}dt$$

$$\frac{4t^3}{3}-\frac{4}{3}\ln\left(t^3+1\right)+C$$

$$yx^{-1/2}=\frac{4x^{3/4}}{3}-\frac{4}{3}\ln\left(x^{3/4}+1\right)+C$$

$$1-\frac{4}{3}\log_e 2=\frac{4}{3}-\frac{4}{3}\log_e 2+C$$

$$\Rightarrow C=-\frac{1}{3}$$

$$y=\frac{4}{3}x^{5/4}-\frac{4}{3}\sqrt{x}\ \ln\left(x^{3/4}+1\right)-\frac{\sqrt{x}}{3}$$

$y(16) = \frac{4}{3} \times 32 - \frac{4}{3} \times 4 \ln 9 - \frac{4}{3}$

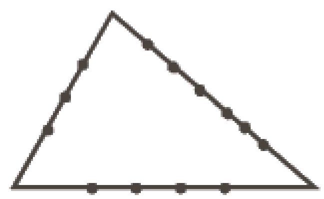
$= \frac{124}{3} - \frac{32}{3} \ln 3 = 4\left(\frac{31}{3} - \frac{8}{3} \ln 3\right)$

11. If the sides AB, BC and CA of a triangle ABC have 3, 5 and 6 interior points respectively, then the total number of triangles that can be constructed using these points as vertices, is equal to:

(a) 364 (b) 240 (c) 333 (d) 360

Ans. (c)

Solution:



Total number of triangles formed

$= {}^{14}C_3 - {}^{3}C_3 - {}^{5}C_3 - {}^{6}C_3$

= 333

12. If x, y, z are in arithmetic progression with common difference d, $x \neq 3d$, and the determinant of the matrix $\begin{bmatrix} 3 & 4\sqrt{2} & x \\ 4 & 5\sqrt{2} & y \\ 5 & k & z \end{bmatrix}$ is zero, then the value of k^2 is

(a) 72 (b) 12 (c) 36 (d) 6

Ans. (a)

Solution:

$$\begin{vmatrix} 3 & 4\sqrt{2} & x \\ 4 & 5\sqrt{2} & y \\ 5 & k & z \end{vmatrix} = 0$$

$R_2 \to R_1 + R_3 - 2R_2$

$$\Rightarrow \begin{vmatrix} 3 & 4\sqrt{2} & x \\ 0 & k - 6\sqrt{2} & 0 \\ 5 & k & z \end{vmatrix} = 0$$

$\Rightarrow \left(k - 6\sqrt{2}\right)(3z - 5x) = 0$

if $3z - 5x = 0 \quad \Rightarrow \quad 3(x + 2d) - 5x = 0$

$\Rightarrow x = 3d$ (Not possible)

$\Rightarrow k = 6\sqrt{2} \quad \Rightarrow \quad k^2 = 72$

13. Let O be the origin. Let $\overrightarrow{OP} = x\hat{i} + y\hat{j} - k$ and $\overrightarrow{OQ} = -\hat{i} + 2\hat{j} + 3xk$, $x, y \in R$, $x > 0$, be such that $\left|\overrightarrow{PQ}\right| = \sqrt{20}$ and the vector $\overrightarrow{OP}$ is perpendicular to $\overrightarrow{OQ}$. If $\overrightarrow{OR} = 3\hat{i} + z\hat{j} - 7k, z \in R$, is coplanar with $\overrightarrow{OP}$ and $\overrightarrow{OQ}$, then the value of $x^2 + y^2 + z^2$ is equal to

(a) 7 (b) 9 (c) 2 (d) 1

Ans. (b)

Solution:

$\overrightarrow{OP} \perp \overrightarrow{OQ}$

$\Rightarrow -x + 2y - 3x = 0$

$\Rightarrow y = 2x$ ...(i)

$\left|\overrightarrow{PQ}\right|^2 = 20$

$\Rightarrow (x+1)^2 + (y-2)^2 + (1+3x)^2 = 20$

$\Rightarrow x = 1$

$\overrightarrow{OP}, \overrightarrow{OQ}, \overrightarrow{QR}$ are coplanar.

$$\Rightarrow \begin{vmatrix} x & y & -1 \\ -1 & 2 & 3x \\ 3 & z & -7 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 1 & 2 & -1 \\ -1 & 2 & 3 \\ 3 & z & -7 \end{vmatrix} = 0$$

$\Rightarrow 1(-14-3z) - 2(7-9) - 1(-z-6) = 0$

$\Rightarrow z = -2$

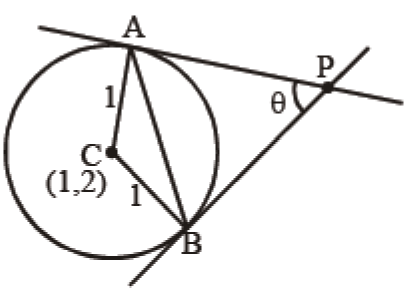
$\therefore\ x^2 + y^2 + z^2 = 1 + 4 + 4 = 9$

14. Two tangents are drawn from a point P to the circle $x^2 + y^2 - 2x - 4y + 4 = 0$, such that the angle between these tangents is $\tan^{-1}\left(\frac{12}{5}\right)$, where $\tan^{-1}\left(\frac{12}{5}\right) \in (0, \pi)$. If the centre of the circle is denoted by C and these tangents touch the circle at points A and B, then the ratio of the areas of ΔPAB and ΔCAB is :

(a) 11 : 4 (b) 9 : 4 (c) 3 : 1 (d) 2 : 1

Ans. (b)

Solution:



$\tan\theta = \frac{12}{5}$

$PA = \cot\frac{\theta}{2}$

$\therefore$ area of $\Delta PAB = \frac{1}{2}(PA)^2 \sin\theta = \frac{1}{2}\cot^2\frac{\theta}{2}\sin\theta$

$= \frac{1}{2}\left(\frac{1+\cos\theta}{1-\cos\theta}\right)\sin\theta$

$= \frac{1}{2}\left(\frac{1+\frac{5}{13}}{1-\frac{5}{13}}\right)\left(\frac{12}{13}\right) = \frac{1}{2}\frac{18}{18}\times\frac{2}{13} = \frac{27}{26}$

area of $\Delta CAB = \frac{1}{2}\sin\theta = \frac{1}{2}\left(\frac{12}{13}\right) = \frac{6}{13}$

$\therefore \frac{\text{area of }\Delta PAB}{\text{area of }\Delta CAB} = \frac{9}{4}$

15. Consider the function $f: R \to R$ defined by $f(x) = \begin{cases} \left(2-\sin\left(\frac{1}{x}\right)\right)|x|, & x \neq 0 \\ 0, & x = 0 \end{cases}$. Then f is:

(a) monotonic on $(-\infty, 0)\cup(0,\infty)$

(b) not monotonic on $(-\infty, 0)$ and $(0,\infty)$

(c) monotonic on $(0,\infty)$ only

(d) monotonic on $(-\infty, 0)$ only

Ans. (b)

Solution:

$$f(x) = \begin{cases} -x\left(2-\sin\left(\frac{1}{x}\right)\right) & x<0 \\ 0 & x=0 \\ x\left(2-\sin\left(\frac{1}{x}\right)\right) \end{cases}$$

$$f'(x) = \begin{cases} -\left(2-\sin\frac{1}{x}\right) - x\left(-\cos\frac{1}{x}\left(-\frac{1}{x^2}\right)\right) & x<0 \\ \left(2-\sin\frac{1}{x}\right) + x\left(\cos\frac{1}{x}\left(-\frac{1}{x^2}\right)\right) & x>0 \end{cases}$$

$$f'(x) = \begin{cases} -2+\sin\frac{1}{x} - \frac{1}{x}\cos\frac{1}{x} & x<0 \\ 2-\sin\frac{1}{x} + \frac{1}{x}\cos\frac{1}{x} & x>0 \end{cases}$$

f ’(x) is an oscillating function which is non-monotonic in $(-\infty, 0)\cup(0,\infty)$.

16. Let L be a tangent line to the parabola $y^2 = 4x - 20$ at (6, 2). If L is also a tangent to the ellipse $\frac{x^2}{2}+\frac{y^2}{b}=1$, then the value of b is equal to:

(a) 11 (b) 14 (c) 16 (d) 20

Ans. (b)

Solution:

Tangent to parabola

$2y = 2(x + 6) - 20$

$\Rightarrow y = x - 4$

Condition of tangency for ellipse.

$16 = 2(1)^2 + b$

$\Rightarrow b = 14$

17. The value of the limit $\lim_{\theta\to 0}\frac{\tan\left(\pi\cos^2\theta\right)}{\sin\left(2\pi\sin^2\theta\right)}$ is equal to:

(a) $-\frac{1}{2}$ (b) $-\frac{1}{4}$ (c) 0 (d) $\frac{1}{4}$

Ans. (a)

Solution:

$$\lim_{\theta\to 0}\frac{\tan\left(\pi\left(1-\sin^2\theta\right)\right)}{\sin\left(2\pi\sin^2\theta\right)}$$

$$=\lim_{\theta\to 0}\frac{-\tan\left(\pi\sin^2\theta\right)}{\sin\left(2\pi\sin^2\theta\right)}$$

$$=\lim_{\theta\to 0}-\left(\frac{\tan\left(\pi\sin^2\theta\right)}{\pi\sin^2\theta}\right)\left(\frac{2\pi\sin^2\theta}{\sin\left(2\pi\sin^2\theta\right)}\right)\times\frac{1}{2}$$

$$=\frac{-1}{2}$$

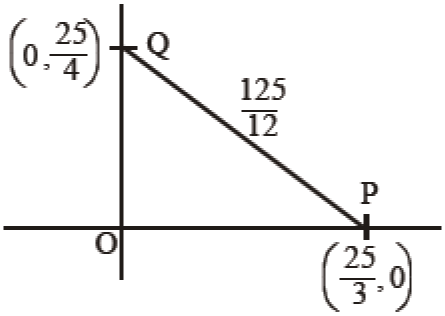
18. Let the tangent to the circle $x^2 + y^2 = 25$ at the point R(3, 4) meet x-axis and y-axis at point P and Q, respectively. If r is the radius of the circle passing through the origin O and having centre at the incentre of the triangle OPQ, then r^2 is equal to

(a) $\frac{529}{64}$ (b) $\frac{125}{72}$ (c) $\frac{625}{72}$ (d) $\frac{585}{66}$

Ans. (c)

Solution:

Tangent to circle $3x + 4y = 25$



OP + OQ + QR = 25

Incentre = $\left(\dfrac{\frac{25}{4}\times\frac{25}{3}}{25},\dfrac{\frac{25}{4}\times\frac{25}{3}}{25}\right)$

= $\left(\frac{25}{12},\frac{25}{12}\right)$

$\therefore\ r^2=2\left(\frac{25}{12}\right)^2=2\times\frac{625}{144}=\frac{625}{72}$

19. If the Boolean expression $(p\wedge q)\odot(p\otimes q)$ is tautology, then $\odot$ and $\otimes$ are respectively given by

(a) $\rightarrow,\rightarrow$ (b) $\wedge,\vee$ (c) $\vee,\rightarrow$ (d) $\wedge,\rightarrow$

Ans. (a)

Solution:

Option (1)

$(p\wedge q)\rightarrow(p\rightarrow q)$

= $\sim(p\wedge q)\vee(\sim p\vee q)$

= $(\sim p\ v\sim q)\ v\ (\sim p\ v\ q)$

= $\sim p\ v\ (\sim q\ v\ q)$

= $\sim p\ v\ t$

= t

Option (2)

$(p\wedge q)\wedge(p\vee q)=(p\wedge q)$ (Not a tautology)

Option (3)

$(p\wedge q)\ v(p\rightarrow q)$

= $(p\wedge q)\ v\ (\sim pvq)$

= $\sim p\ v\ q$

Option (4)

= $(p\wedge q)\wedge(\sim p\ v\ q)$

= $p\wedge q$ (Not a tautology)

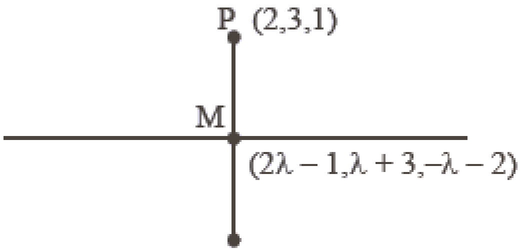
20. If the equation of plane passing through the mirror image of a point (2, 3, 1) with respect to line $\frac{x+1}{2}=\frac{y-3}{1}=\frac{z+2}{-1}$ and containing the line $\frac{x-2}{3}=\frac{1-y}{2}=\frac{z+1}{1}$ is $\alpha x+\beta\gamma+\gamma z=24$, then $\alpha+\beta+\gamma$ is equal to:

(a) 20 (b) 19 (c) 18 (d) 21

Ans. (b)

Solution:

Line $\frac{x+1}{2}=\frac{y-3}{1}=\frac{z+2}{-1}$



$\overrightarrow{PM}=(2\lambda-3,\lambda,-\lambda-3)$

$\overrightarrow{PM}\perp(2\hat{i}+\hat{j}-k)$

$4\lambda-6+\lambda+\lambda+3=0 \Rightarrow \lambda=\frac{1}{2}$

$\therefore\ M\equiv\left(0,\frac{7}{2},\frac{-5}{2}\right)$

$\therefore$ Reflection (−2, 4, − 6)

Plane: $\begin{vmatrix} x-2 & y-1 & z+1 \\ 3 & -2 & 1 \\ 4 & -3 & 5 \end{vmatrix}=0$

$\Rightarrow (x-2)(-10+3)-(y-1)(15-4)+(z+1)(-1)=0$

$\Rightarrow -7x+14-11y+11-z-1=0$

$\Rightarrow 7x+11y+z=24$

$\therefore\ \alpha=7,\beta=11,\gamma=1$

$\alpha+\beta+\gamma=19$

Integer Type

1. If 1, $\log_{10}(4x - 2)$ and $\log_{10}\left(4^x + \frac{18}{5}\right)$ are in arithmetic progression for a real number x, then the value of the determinant $\begin{vmatrix} 2\left(x-\frac{1}{2}\right) & x-1 & x^2 \\ 1 & 0 & x \\ x & 1 & 0 \end{vmatrix}$ is equal to:

Ans. 2

Solution:

$$2\log_{10}\left(4^x - 2\right) = 1 + \log_{10}\left(4^x + \frac{18}{5}\right)$$

$$\left(4^x - 2\right)^2 = 10\left(4^x + \frac{18}{5}\right)$$

$$\left(4^x\right)^2 + 4 - 4\left(4^x\right) - 32 = 0$$

$4^x = 16$

x = 2

$$\begin{vmatrix} 3 & 1 & 4 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \end{vmatrix} = 3(-2) - 1(0-4) + 4(1)$$

= –6 + 4 + 4 = 2

2. Let f: [–1, 1] $\to R$ be defined as f(x) = $ax^2 + bx + c$ for all $x \in [-1, 1]$, where a, b, c $\in R$ such that f(–1) = 2, f'(–1) = 1 and for $x \in (-1, 1)$ the maximum value of f''(x) is $\frac{1}{2}$. If f(x) $\le \alpha$, $x \in [-1, 1]$, then the least value of α is equal to ________ .

Ans. 5

Solution:

$f : [-1, 1] \to R$

$f(x) = ax^2 + bx + c$

f(–1) = a – b + c = 2 ...(1)

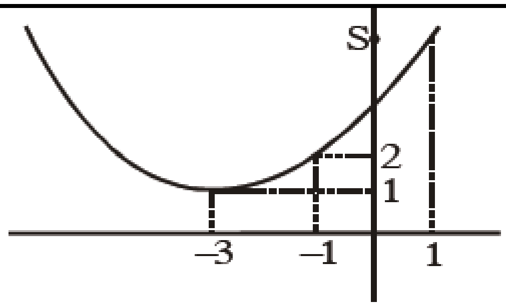
f'(–1) = –2a + b = 1 ...(2)

f''(x) = 2a

$\Rightarrow$ Max. value of f''(x) = 2a = $\frac{1}{2}$

$$\Rightarrow a = \frac{1}{4};\ b = \frac{3}{2};\ c = \frac{13}{4}$$

$$\therefore f(x) = \frac{x^2}{4} + \frac{3}{2}x + \frac{13}{4}$$



For, $x \in [-1, 1] \Rightarrow 2 \le f(x) \le 5$

$\therefore$ Least value of α is 5

3. Let f: [–3, 1] $\to R$ be given as f(x) = $\begin{cases} \min\{(x+6), x^2\}, & -3 \le x \le 0 \\ \max\{\sqrt{x}, x^2\}, & 0 \le x \le 1. \end{cases}$

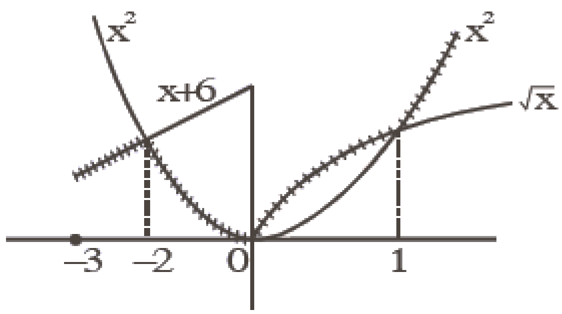
If the area bounded by y = f(x) and x-axis is A, then the value of 6A is equal to ________ .

Ans. 41

Solution:

f: [–3, 1] $\to R$

$$f(x) = \begin{cases} \min\{(x+6), x^2\}, & -3 \le x \le 0 \\ \max\{\sqrt{x}, x^2\}, & 0 \le x \le 1 \end{cases}$$



are bounded by y = f(x) and x-axis

$$= \int_{-3}^{-2} (x+6)dx + \int_{-2}^{0} x^2 dx + \int_{0}^{1} \sqrt{x} dx$$

$$A = \frac{41}{6}$$

6A = 41

4. Let tan α, tan β and tan γ; $\alpha, \beta, \gamma \ne \frac{(2n-1)\pi}{2}$, $n \in N$ be the slopes of three line segments OA, OB and OC, respectively, where O is origin. If circumcentre of ΔABC coincides with origin and its orthocentre lies on y-axis, then the value of $\left(\frac{\cos 3\alpha + \cos 3\beta + \cos 3\gamma}{\cos\alpha \cos\beta \cos\gamma}\right)^2$ is equal to:

Ans. 144

Solution:

Since orthocentre both lies on y-axis

$\Rightarrow$ Centroid also lies on y-axis

$\Rightarrow \sum \cos\alpha = 0$

$\cos\alpha + \cos\beta + \cos\gamma = 0$

$\Rightarrow \cos^3\alpha + \cos^3\beta + \cos^3\gamma = 3\cos\alpha\cos\beta\cos\gamma$

$\therefore \dfrac{\cos 3\alpha + \cos 3\beta + \cos 3\gamma}{\cos\alpha\cos\beta\cos\gamma}$

$= \dfrac{4\left(\cos^3\alpha + \cos^3\beta + \cos^3\gamma\right) - 3\left(\cos\alpha + \cos\beta + \cos\gamma\right)}{\cos\alpha\cos\beta\cos\gamma}$

= 12

5. Consider a set of 3n numbers having variance 4. In this set, the mean of first 2n numbers is 6 and the mean of the remaining n numbers is 3. A new set is constructed by adding 1 into each of first 2n numbers, and subtracting 1 from each of the remaining n numbers. If the variance of the new set is k, then 9k is equal to ________ .

Ans. 68

Solution:

Let number be $a_1, a_2, a_3, \ldots a_{2n}, b_1, b_2, b_3, \ldots b_n$ $\sigma^2 = \dfrac{\sum a^2 + \sum b^2}{3n} - (5)^2$

$\Rightarrow \sum a^2 + \sum b^2 = 87n$

Now, distribution becomes

$a_1 + 1, a_2 + 1, a_3 + 1, \ldots a_{2n} + 1, b_1 - 1, b_2 - 1 \ldots b_n - 1$

Variance

$= \dfrac{\sum(a+1)^2 + \sum(b-1)^2}{3n} - \left(\dfrac{12n + 2n + 3n - n}{3n}\right)^2$

$= \dfrac{\left(\sum a^2 + 2n + 2\sum a\right) + \left(\sum b^2 + n - 2\sum b\right)}{3n}$

$= \dfrac{\left(\sum a^2 + 2n + 2\sum a\right) + \left(\sum b^2 + n - 2\sum b\right)}{3n} - \left(\dfrac{16}{3}\right)^2$

$= \dfrac{87n + 3n + 2(12n) - 2(3n)}{3n} - \left(\dfrac{16}{3}\right)^2$

$\Rightarrow k = \dfrac{108}{3} - \left(\dfrac{16}{5}\right)^2$

$\Rightarrow 9k = 3(108) - (16)^2 = 324 - 256 = 68$

6. Let the coefficients of third, fourth and fifth terms in the expansion of $\left(x + \dfrac{a}{x^2}\right)^n, x \neq 0,$ be in the ratio 12: 8: 3. Then the term independent of x in the expansion, is equal to ________ .

Ans. 4

Solution:

$T_{r+1} = {}^nC_r (x)^{n-r}\left(\frac{a}{x^2}\right)^r$

$= {}^nC_r\, a^r x^{n-3r}$

${}^nC_2\, a^2 : {}^nC_3\, a^3 : {}^nC_4\, a^4 = 12 : 8 : 3$

After solving

$n = 6, a = \frac{1}{2}$

For term independent of 'x' $\Rightarrow$ $n = 3r$

r = 2

$\therefore$ Coefficient is ${}^6C_2\left(\frac{1}{2}\right)^2 = \frac{15}{4}$

Nearest integer is 4.

7. Let A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ such that AB = B and a + d = 2021, then the value of ad – bc is equal to __________ .

Ans. 2020

Solution:

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, B = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$

AB = B

$\Rightarrow (A - I)B = O$

$\Rightarrow |A - I| = O$, since $B \neq O$

$\begin{vmatrix} (a-1) & b \\ c & (d-1) \end{vmatrix} = 0$

ad – bc = 2020

8. Let $\vec{x}$ be a vector in the plane containing vectors $\vec{a} = 2\hat{i} - \hat{j} + k$ and $\vec{b} = \hat{i} + 2\hat{j} - k$. If the vector $\vec{x}$ is perpendicular to $\left(3\hat{i} + 2\hat{j} - k\right)$ and its projection on $\vec{a}$ is $\frac{17\sqrt{6}}{2}$, then the value of $|\vec{x}|^2$ is equal to____ .

Ans. 486

Solution:

Let $\vec{x} = \lambda\vec{a} + \mu\vec{b}$ $\quad$ (λ and μ are scalars)

$\vec{x} = \hat{i}(2\lambda + \mu) + \hat{j}(2\mu - \lambda) + k(\lambda - \mu)$

Since $\vec{x}.\left(3\hat{i} + 2\hat{j} - k\right) = 0$

$3\lambda + 8\mu = 0$ $\quad$ ...(1)

Also projection of $\vec{x}$ on $\vec{a}$ is $\frac{17\sqrt{6}}{2}$

$$\frac{\vec{x}.\vec{a}}{|\vec{a}|}=\frac{17\sqrt{6}}{2}$$

$6\lambda-\mu=51$ ...(2)

From (1) and (2)

$\lambda=8, \mu=-3$

$\vec{x}=13\hat{i}-14\hat{j}+11k$

$|\vec{x}|^2=486$

9. Let $I_n=\int_1^e x^{19}(\log|x|)^n dx$, where $n\in N$. If $(20)I_{10}=\alpha I_9+\beta I_8$, for natural numbers α and β, then $\alpha-\beta$ equal to ________ .

Ans. 1

Solution:

$$I_n=\int_1^e x^{19}(\log|x|)^n dx$$

$$I_n=\left|(\log|x|)^{19}\frac{x^{20}}{20}\right|_1^e-\int n(\log|x|)^{n-1}.\frac{1}{x}.\frac{x^{20}}{20}dx$$

$20I_n=e^{20}-nI_{n-1}$

$\therefore\ 20I_{10}=e^{20}-10I_9$

$20I_9=e^{20}-9I_8$

$\Rightarrow\ 20I_{10}=10I_9+9I_8$

$\alpha=10, \beta=9$

10. Let P be an arbitrary point having sum of the squares of the distance from the planes x + y + z = 0, lx – nz = 0 and x – 2y + z = 0, equal to 9. If the locus of the point P is $x^2 + y^2 + z^2 = 9$, then the value of l – n is equal to __________.

Ans. 0

Solution:

Let point P is (α,β,γ)

$$\left(\frac{\alpha+\beta+\gamma}{\sqrt{3}}\right)^2+\left(\frac{\ell\alpha-ny}{\sqrt{\ell^2+n^2}}\right)^2+\left(\frac{\alpha-2\beta+\gamma}{\sqrt{6}}\right)^2=9$$

Locus is

$$\frac{(x+y+z)^2}{3}+\frac{(\ell x-nz)^2}{\ell^2+n^2}+\frac{(x-2y+z)^2}{6}=9$$

$$x^2\left(\frac{1}{2}+\frac{\ell^2}{\ell^2+n^2}\right)+y^2+z^2\left(\frac{1}{2}+\frac{n^2}{\ell^2+n^2}\right)+2zx\left(\frac{1}{2}-\frac{\ell n}{\ell^2+n^2}\right)-9=0$$

Since its given that $x^2 + y^2 + z^2 = 9$

After solving $\ell=n$