JEE-Main Mathematics Questions Paper 16.3.2021 Shift-2

1. The maximum value of

$$f(x)=\begin{vmatrix} \sin^2 x & 1+\cos^2 x & \cos 2x \\ 1+\sin^2 x & \cos^2 x & \cos 2x \\ \sin^2 x & \cos^2 x & \sin 2x \end{vmatrix}, x\in R \text{ is:}$$

(a) $\sqrt{7}$ (b) $\frac{3}{4}$ (c) $\sqrt{5}$ (d) 5

Ans. (c)

Solution:

$C_1+C_2\to C_1$

$$\begin{vmatrix} 2 & 1+\cos^2 x & \cos 2x \\ 2 & \cos^2 x & \cos 2x \\ 1 & \cos^2 x & \sin 2x \end{vmatrix}$$

$R_1-R_2\to R_1$

$$\begin{vmatrix} 0 & 1 & 0 \\ 2 & \cos^2 x & \cos 2x \\ 1 & \cos^2 x & \sin 2x \end{vmatrix}$$

Open w. r. t. R_1

$-2(\sin 2x - \cos 2x)$

$\cos 2x - 2 \sin 2x = f(x)$

$f(x)|_{max}=\sqrt{1+4}=\sqrt{5}$

2. Let A denote the event that a 6-digit integer formed by 0, 1, 2, 3, 4, 5, 6 without repetitions, be divisible by 3. Then probability of event A is equal to:

(a) $\frac{9}{56}$ (b) $\frac{4}{9}$ (c) $\frac{3}{7}$ (d) $\frac{11}{27}$

Ans. (b)

Solution:

Total cases:

6. 6. 5. 4. 3. 2

n(s) = 6. 6!

Favourable cases:

Number divisible by 3 ≡

Sum of digits must be divisible by 3

Case-I

1, 2, 3, 4, 5, 6

Number of ways = 6!

Case-II

0, 1, 2, 4, 5, 6

Number of ways = 5. 5!

Case-III

0, 1, 2, 3, 4, 5

Number of ways = 5. 5!

Case-III

0, 1, 2, 3, 4, 5

Number of ways = 5.5!

n(favourable) = 6! + 2. 5. 5!

$$P=\frac{6!+2.5.\,5!}{6.6!}=\frac{4}{9}$$

3. Let $\alpha\in R$ be such that the function

$$f(x)=\begin{cases}\dfrac{\cos^{-1}(1-\{x\}^2)\sin^{-1}(1-\{x\})}{\{x\}-\{x\}^3}, & x\\ \alpha, & x=0\end{cases}$$

is continuous at x = 0, where {x} = x – [x], [x] is the greatest integer less than or equal to x. Then:

(a) $\alpha=\frac{\pi}{\sqrt{2}}$ (b) $\alpha=0$ (c) no such α exists (d) $\alpha=\frac{\pi}{4}$

Ans. (c)

Solution:

$$\lim_{x\to 0^+} f(x)=f(0)=\lim_{x\to 0^-}(x)$$

$$\lim_{x\to 0^+}\frac{\cos^{-1}\left(1-x^2\right).\sin^{-1}(1-x)}{x(1-x)(1+x)}$$

$$\lim_{x\to 0^+}\frac{\cos^{-1}\left(1-x^2\right)}{x.1.1}.\frac{\pi}{2}$$

Let $1-x^2=\cos\theta$

$$\frac{\pi}{2}\lim_{x\to 0^+}\frac{\theta}{\sqrt{1-\cos\theta}}$$

$$\frac{\pi}{2}\lim_{\theta\to 0^+}\frac{\theta}{\sqrt{2}\sin\frac{\theta}{2}}=\frac{\pi}{\sqrt{2}}$$

Now, $\lim_{x\to 0^-}\dfrac{\cos^{-1}\left(1-(1+x)^2\right)\sin^{-1}(-x)}{(1+x)-(1+x)^3}$

$$\lim_{x\to 0^-}\frac{\frac{\pi}{2}\left(-\sin^{-1}x\right)}{(1+x)(2+x)(-x)}$$

$$\lim_{x\to 0^-}\frac{\frac{\pi}{2}}{1.2}.\frac{\sin^{-1}x}{x}=\frac{\pi}{4}$$

$\Rightarrow$ RHL $\neq$ LHL

Function can’t be continuous

$\Rightarrow$ No value of α exist

4. If (x, y, z) be an arbitrary point lying on a plane P which passes through the point (42, 0, 0), (0, 42, 0) and (0, 0, 42), then the value of expression

$$3+\frac{x-11}{(y-19)^2(z-12)^2}+\frac{y-19}{(x-11)^2(z-12)^2}+\frac{z-12}{(x-11)^2(y-19)^2}-\frac{x+y+z}{14(x-11)(y-19)(z-12)}$$

(a) 0 (b) 3 (c) 39 (d) – 45

Ans. (b)

Solution:

Plane passing through (42, 0, 0), (0, 42, 0), (0, 0, 42)

From intercept from, equation of plane is x + y + z = 42

$\Rightarrow (x-1)+(y-19)+(z-12)=0$

let a = x – 11, b = y – 10, c = z – 12

a + b + c = 0

Now, given expression is

$$3+\frac{a}{b^2c^2}+\frac{b}{a^2c^2}+\frac{c}{a^2b^2}-\frac{42}{14abc}$$

$$3+\frac{a^3+b^3+c^3-3abc}{a^2b^2c^2}$$

If a + b + c = 0

$\Rightarrow a^3 + b^3 + c^3 = 3\ abc$

$\Rightarrow 3$

5. Consider the integral $I=\int_0^{10}\frac{[x]e^{[x]}}{e^{x-1}}dx,$

where [x] denotes the greatest integer less than or equal to x. Then the value of I is equal to:

(a) 9(e – 1) (b) 45(e + 1) (c) 45(e – 1) (d) 9(e + 1)

Ans. (c)

Solution:

$$I=\int_0^{10}[x].e^{[x]-x+1}$$

$$I=\int_0^1 0dx+\int_1^2 1.e^{2-x}+\int_2^3 2.e^{3-x}+....+\int_9^{10} 9.e^{10-x}dx$$

$$\Rightarrow I=\sum_{n=0}^{9}\int_n^{n+1} n.e^{n+1-x}\,dx$$

$$= -\sum_{n=0}^{9} n\left(e^{n+1-x}\right)_n^{n+1}$$

$$= -\sum_{n=0}^{9} n.\left(e^0-e^1\right)$$

$$= (e-1)\sum_{n=0}^{9} n$$

$= (e-1).\frac{9-10}{2}$

= $45(e - 1)$

6. Let C be the locus of the mirror image of a point on the parabola $y^2 = 4x$ with respect to the line y = x. Then the equation of tangent to C at P(2, 1) is:

(a) x – y = 1 (b) 2x + y = 5 (c) x + 3y = 5 (d) x + 2y = 4

Ans. (a)

Solution:

Given $y^2 = 4x$

Mirror image on y = x $\Rightarrow$ $C : x^2 = 4y$

$2x = 4.\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{x}{2}$

$\left.\frac{dy}{dx}\right|_{P(2,1)} = \frac{2}{2} = 1$

Equation of tangent at (2, 1)

$\Rightarrow y - 1 = 1(x-2)$

$\Rightarrow x - y = 1$

7. If y = y(x) is the solution of the differential equation $\frac{dy}{dx} + (\tan x)y = \sin x, 0 \le x \le \frac{\pi}{3}$, with y(0) = 0, then $y\left(\frac{\pi}{4}\right)$ equal to:

(a) $\frac{1}{4}\log_e 2$ (b) $\left(\frac{1}{2\sqrt{2}}\right)\log_e 2$ (c) $\log_e 2$ (d) $\frac{1}{2}\log_e 2$

Ans. (b)

Solution:

$\frac{dy}{dx} + (\tan x)y = \sin x;\ 0 \le x \le \frac{\pi}{3}$

I.F. = $e^{\int \tan x\, dx} = e^{\ell n \sec x} = \sec x$

$y \sec x = \int \tan x\, dx$

$y \sec x = \int \tan x\, dx$

$y \sec x = \ell n|\sec x| + C$

$x = 0,\ y = 0 \Rightarrow\ \therefore\ c = 0$

$y \sec x = \ell n\,|\sec x|$

$y = \cos x.\ \ell n\,|\sec x|$

$\left.y\right|_{x=\frac{\pi}{4}} = \left(\frac{1}{\sqrt{2}}\right).\ \ell n\sqrt{2}$

$\left.y\right|_{x=\frac{\pi}{4}} = \frac{1}{2\sqrt{2}}\log_e 2$

8. Let A = {2, 3, 4, 5,..., 30} and '$\simeq$' be an equivalence relation on $A \times A$, defined by $(a, b) \simeq (c, d)$, if and only if ad = bc. Then the number of ordered pairs which satisfy this equivalence relation with ordered pair (4, 3) is equal to:

(a) 5 (b) 6 (c) 8 (d) 7

Ans. (d)

Solution:

A = {2, 3, 4, 5, ..., 30}

$(a, b) \simeq (c, d) \quad \Rightarrow \quad ad = bc$

$(4, 3) \simeq (c, d) \quad \Rightarrow \quad 4d = 3c$

$\Rightarrow \frac{4}{3} = \frac{c}{d}$

$\frac{c}{d} = \frac{4}{3}$ & $c, d \in \{2, 3, ...,30\}$

(c, d) = {(4, 3), (8, 6), (12, 9), (16, 12), (20, 15), (24, 18), (28, 21)}

No. of ordered pair = 7

9. Let the lengths of intercepts on x-axis and y-axis made by the circle $x^2 + y^2 + ax = 2ay + c = 0$, (a < 0) be $2\sqrt{2}$ and $2\sqrt{5}$, respectively. Then the shortest distance from origin to a tangent to this circle which is perpendicular to the line x +2y = 0, is equal to:

(a) $\sqrt{11}$ (b) $\sqrt{7}$ (c) $\sqrt{6}$ (d) $\sqrt{10}$

Ans. (c)

Solution:

$x^2 + y^2 + ax + 2ay + c = 0$

$2\sqrt{g^2 - c} = 2\sqrt{\frac{a^2}{4} - c} = 2\sqrt{2}$

$\Rightarrow \frac{a^2}{4} - c = 2 \qquad ...(1)$

$2\sqrt{f^2 - c} = 2\sqrt{a^2 - c} = 2\sqrt{5}$

$\Rightarrow a^2 - c = 5 \qquad ...(2)$

(1) & (2)

$\frac{3a^2}{4} = 3 \Rightarrow a = -2 \quad (a < 0)$

$\therefore \ c = -1$

Circle $\Rightarrow x^2 + y^2 - 2x - 4y - 1 = 0$

$\Rightarrow (x-1)^2 + (y-2)^2 = 6$

Given x + 2y = 0 $\quad \Rightarrow \quad m = -\frac{1}{2}$

$m_{tangent} = 2$

Equation of tangent

$\Rightarrow (y-2) = 2(x-1) \pm \sqrt{6}\sqrt{1+4}$

$\Rightarrow 2x - y \pm \sqrt{30} = 0$

Perpendicular distance from (0, 0) = $\left|\frac{\pm\sqrt{30}}{\sqrt{4+1}}\right| = \sqrt{6}$

10. The least value of $|z|$ where z is complex number which satisfy the inequality

$$\exp\left(\frac{(|z|+3)(|z|-1)}{|z|+1}\log_e 2\right) \ge \log_{\sqrt{2}}\left|5\sqrt{7}+9i\right|,\ i=\sqrt{-1}, \text{is equal to:}$$

(a) 3 (b) $\sqrt{5}$ (c) 2 (d) 8

Ans. (a)

Solution:

$$\exp\left(\frac{(|z|+3)(|z|-1)}{||z|+1|}\ell n2\right) \ge \log_{\sqrt{2}}\left|5\sqrt{7}+9i\right|$$

$$\Rightarrow 2^{\frac{(|z|+3)(|z|-1)}{(|z|+1)}} \ge \log_{\sqrt{2}}(16)$$

$$\Rightarrow 2^{\frac{(|z|+3)(|z|-1)}{(|z|+1)}} \ge 2^3$$

$$\Rightarrow \frac{(|z|+3)(|z|-1)}{(|z|+1)} \ge 3$$

$$\Rightarrow (|z|+3)(|z|-1) \ge 3(|z|+1)$$

$$|z|^2 + 2|z| - 3 \ge 3|z| + 3$$

$$\Rightarrow |z|^2 + |z| - 6 \ge 0$$

$$\Rightarrow (|z|-3)(|z|+2) \ge 0 \ \Rightarrow |z|-3 \ge 0$$

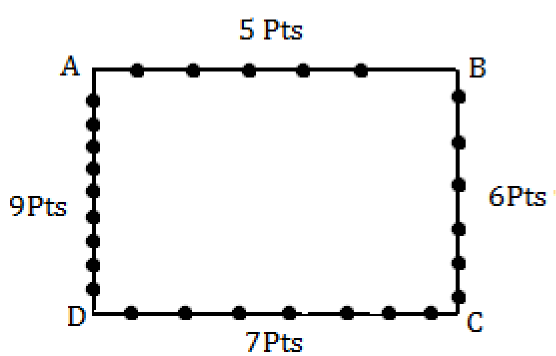
$$\Rightarrow |z| \ge 3 \quad \Rightarrow \quad |z|_{min} = 3$$

11. Consider a rectangle ABCD having 5, 7, 6, 9 points in the interior of the line segments AB, CD, BC, DA respectively. Let α be the number of triangles having these points from different sides as vertices and β be the number of quadrilaterals having these points from different sides as vertices. Then $(\beta-\alpha)$ is equal to:

(a) 795 (b) 1173 (c) 1890 (d) 717

Ans. (d)

Solution:



α = Number of triangles

α = 5.6.7 + 5.7.9 + 5. 6. 9 + 6. 7. 9

= 210 + 315 + 270 + 378

= 1173

β = Number of Quadrilateral

$\beta = 5.\,6.\,7.\,9 = 1890$

$\beta - \alpha = 1890 - 1173 = 717$

12. If the point of intersections of the ellipse $\frac{x^2}{16}+\frac{y^2}{b^2}=1$ and the circle $x^2 + y^2 = 4b$, b > 4 lie on the curve $y^2 = 3x^2$, then b is equal to:

(a) 12 (b) 5 (c) 6 (d) 10

Ans. (a)

Solution:

$y^2 = 3x^2$

and $x^2 + y^2 = 4b$

Solve both we get

so $x^2 = b$

$$\frac{x^2}{16}+\frac{3x^2}{b^2}=1$$

$$\frac{b}{16}+\frac{3}{b}=1$$

$b^2 - 16b + 48 = 0$

$(b - 12)(b - 4) = 0$

b = 12, b > 4

13. Given that the inverse trigonometric functions take principal values only. Then, the number of real values of x which satisfy $\sin^{-1}\left(\frac{3x}{5}\right)+\sin^{-1}\left(\frac{4x}{5}\right)=\sin^{-1}x$ is equal to:

(a) 2 (b) 1 (c) 3 (d) 0

Ans. (c)

Solution:

$$\sin^{-1}\frac{3x}{5}+\sin^{-1}\frac{4x}{5}=\sin^{-1}x$$

$$\sin^{-1}\left(\frac{3x}{5}\sqrt{1-\frac{16x^2}{25}}+\frac{4x}{5}\sqrt{1-\frac{9x^2}{25}}\right)=\sin^{-1}x$$

$$\frac{3x}{5}\sqrt{1-\frac{16x^2}{25}}+\frac{4x}{5}\sqrt{1-\frac{9x^2}{25}}=x$$

$$x=0,\ 3\sqrt{25-16x^2}+4\sqrt{25-9x^2}=25$$

$4\sqrt{25-9x^2}=25-3\sqrt{25-16x^2}$ squaring we get

$$16\left(25-9x^2\right)=625+9\left(25-16x^2\right)-150\sqrt{25-16x^2}$$

$400 = 625 + 225 - 150\sqrt{25-16x^2}$

$$\sqrt{25-16x^2}=3 \Rightarrow 25-16x^2=9$$

$$\Rightarrow x^2=1$$

Put x = 0, 1, – 1 in the original equation

We see that all values satisfy the original equation.

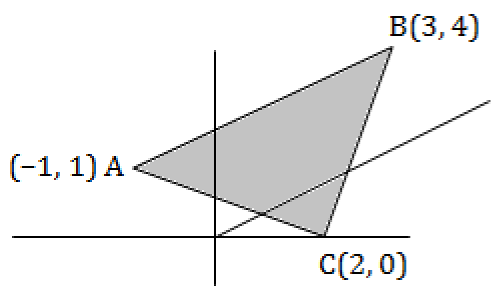
Number of solution = 3

14. Let A(−1, 1), B(3, 4) and C(2, 0) be given three points. A line y = mx, m > 0, intersects lines AC and BC at point P and Q respectively. Let A_1 and A_2 be the areas of ΔABC and ΔPQC respectively, such that $A_1 = 3A^2$, then the value of m is equal to:

(a) $\frac{4}{15}$ (b) 1 (c) 2 (d) 3

Ans. (b)

Solution:



$P \equiv (x_1, mx_1)$

$Q \equiv (x_2, mx_2)$

$$A_1 = \frac{1}{2}\begin{vmatrix} 3 & 4 & 1 \\ 2 & 0 & 1 \\ -1 & 1 & 1 \end{vmatrix} = \frac{13}{2}$$

$$A_2 = \frac{1}{2}\begin{vmatrix} x_1 & mx_1 & 1 \\ x_2 & mx_2 & 1 \\ 2 & 0 & 1 \end{vmatrix}$$

$$A_2 = \frac{1}{2}\left|2(mx_1 - mx_2)\right| = m|x_1 - x_2|$$

$$A_1 = 3A_2 \Rightarrow \frac{13}{2} = 3m|x_1 - x_2|$$

$$\Rightarrow |x_1 - x_2| = \frac{16}{6m}$$

AC : x + 3y = 2

BC : y = 4x – 8

P: x + 3y = 2 & y = mx $\Rightarrow x_1 = \frac{2}{1+3m}$

Q: y = 4x – 8 & y = mx $\Rightarrow x_2 = \frac{8}{4-m}$

$$|x_1 - x_2| = \left|\frac{2}{1+3m} - \frac{8}{4-m}\right|$$

$$= \left|\frac{-26m}{(1+3m)(4-m)}\right| = \frac{26m}{(3m+1)|m-4|}$$

$= \frac{26m}{(3m+1)(4-m)}$

$|x_1 - x_2| = \frac{13}{6m}$

$\frac{26m}{(3m+1)(4-m)} = \frac{13}{6m}$

$\Rightarrow 12m^2 = -(3m+1)(m-4)$

$\Rightarrow 12m^2 = -(3m^2-1)(m-4)$

$\Rightarrow 15m^2 - 11m - 4 = 0$

$\Rightarrow 15m^2 - 15m + 4m - 4 = 0$

$\Rightarrow (15m+4)(m-1) = 0$

$\Rightarrow m = 1$

15. Let f be a real valued function, defined on R – {–1, 1} and given by

$f(x) = 3\log_e\left|\frac{x-1}{x+1}\right| - \frac{2}{x-1}.$

Then in which of the following intervals, function f(x) is increasing?

(a) $(-\infty, -1) \cup \left(\left[\frac{1}{2}, \infty\right) - \{1\}\right)$

(b) $(-\infty, \infty) - \{-1, 1\}$

(c) $\left(-1, \frac{1}{2}\right]$

(d) $\left(-\infty, \frac{1}{2}\right] - \{-1\}$

Ans. (a)

Solution:

$f(x) = 3\ell n(x-1) - 3\ell n(x+1) - \frac{2}{x-1}$

$f'(x) = \frac{3}{x-1} - \frac{3}{x+1} + \frac{2}{(x-1)^2}$

$f'(x) = \frac{4(2x-1)}{(x-1)^2(x+1)}$

$f'(x) \geq 0$

$\Rightarrow x \in (-\infty, -1) \cup \left[\frac{1}{2}, 1\right) \cup (1, \infty)$

16. Let $f: S \to S$ where S = $(0, \infty)$ be a twice differentiable function such that f(x + 1) = xf(x).

If $g: S \to R$ be defined as g(x) = $\log_e f(x)$, then the value of $|g''(5) - g''(1)|$ is equal to:

(a) $\frac{205}{144}$ (b) $\frac{197}{144}$ (c) $\frac{187}{144}$ (d) 1

Ans. (a)

Solution:

lnf(x + 1) = ln(xf(x))

lnf(x + 1) = lnx + lnf(x)

$\Rightarrow g(x+1) = \ln x + g(x)$

$\Rightarrow g(x+1) - g(x) = \ln x$

$\Rightarrow g''(x+1) - g''(x) = -\frac{1}{x^2}$

Put x = 1, 2, 3, 4

$g''(2) - g''(1) = -\frac{1}{t^2}$ ...(1)

$g''(3) - g''(2) = -\frac{1}{2^2}$ ...(2)

$g''(4) - g''(3) = -\frac{1}{3^2}$ ...(3)

$g''(5) - g''(4) = -\frac{1}{4^2}$ ...(4)

Add all the equation we get

$g''(5) - g''(1) = -\frac{1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2}$

$|g''(5) - g''(1)| = \frac{205}{144}$

17. Let P(x) = x^2 + bx + c be a quadratic polynomial with real coefficients such that $\int_0^1 P(x)dx = 1$ and P(x) leaves remainder 5 when it is divided by (x – 2). Then the value of 9(b + c) is equal to:

(a) 9 (b) 15 (c) 7 (d) 11

Ans. (c)

Solution:

$\int_0^1 (x^2 + bx + c)dx = 1$

$\frac{1}{3} + \frac{b}{2} + c = 1 \quad \Rightarrow \quad \frac{b}{2} + c = \frac{2}{3}$

3b + 6c = 4 ...(1)

P(2) = 5

4 + 2b + c = 5

2b + c = 1 ...(2)

From (1) & (2)

b = $\frac{2}{9}$ & c = $\frac{5}{9}$

9(b + c) = 7

18. If the foot of the perpendicular from point (4, 3, 8) on the line L_1: $\frac{x-a}{l}=\frac{y-2}{3}=\frac{z-b}{4}$, $l \neq 0$ is (3, 5, 7), then the shortest distance between the line L_1 and line L_2: $\frac{x-2}{3}=\frac{y-4}{4}=\frac{z-5}{5}$ is equal to:

(a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{6}}$ (c) $\sqrt{\frac{2}{3}}$ (d) $\frac{1}{\sqrt{3}}$

Ans. (b)

Solution:

(3, 5, 7) satisfy the line L_1

$$\frac{3-a}{\ell}=\frac{5-2}{3}=\frac{7-b}{4}$$

$$\frac{3-a}{\ell}=1 \quad \& \quad \frac{7-b}{4}=1$$

$a+\ell=3$ ...(1) & b = 3 ....(2)

$\vec{v}_1 = <4, 3, 8> - <3, 5, 7>$

$\vec{v}_1 = <1, -2, 1>$

$\vec{v}_2 = <\ell, 3, 4>$

$\vec{v}_1 . \vec{v}_2 = 0 \Rightarrow \ell-6+4=0 \Rightarrow \ell=2$

$a+\ell=3 \Rightarrow a=1$

$$L_1: \frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$$

$$L_2: \frac{x-2}{3}=\frac{y-4}{4}=\frac{z-5}{5}$$

A = <1, 2, 3>

B = <2, 4, 5>

$\overrightarrow{AB} = <1, 2, 2>$

$\vec{p} = 2\hat{i}+3\hat{j}+4\hat{k}$

$\vec{q} = 3\hat{i}+4\hat{j}+5\hat{k}$

$\vec{p}\times\vec{q} = -\hat{i}+2\hat{j}-\hat{k}$

Shortest distance = $\left|\frac{\overrightarrow{AB}.(\vec{p}\times\vec{q})}{|\vec{p}\times\vec{q}|}\right| = \frac{1}{\sqrt{6}}$

19. Let C_1 be the curve obtained by the solution of differential equation $2xy\frac{dy}{dx}=y^2-x^2,\ x>0$.

Let the curve C_2 be the solution of $\frac{2xy}{x^2-y^2}=\frac{dy}{dx}$. If both the curves pass through (1, 1), then the area enclosed by the curve C_1 and C_2 is equal to:

(a) $\pi-1$ (b) $\frac{\pi}{2}-1$ (c) $\pi+1$ (d) $\frac{\pi}{4}+1$

Ans. (b)

Solution:

$$\frac{dy}{dx}=\frac{y^2-x^2}{2xy}, x\in(0\ \infty)$$

put y = vx

$$x\frac{dv}{dx}+v=\frac{v^2-1}{2v}$$

$$\frac{2v}{v^2+1}\,dv=-\frac{dx}{x}$$

Integrate,

$$\ln(v^2+1)=-\ln x+C$$

$$\ln\left(\frac{y^2}{x^2}+1\right)=-\ln x+C$$

put x = 1, y = 1, C = ln 2

$$\ln\left(\frac{y^2}{x^2}+1\right)=-\ln x+\ln 2$$

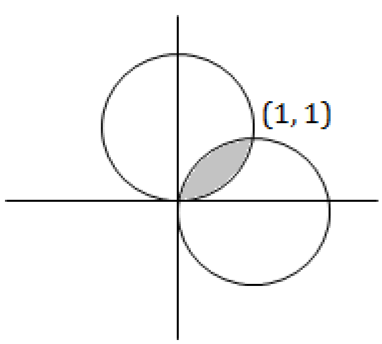
$\Rightarrow x^2+y^2-2x=0$ (Curve C_1)

Similarly,

$$\frac{dy}{dx}=\frac{2xy}{x^2-y^2}$$

Put y = vx

$x^2+y^2-2y=0$



required area $=2\int_0^1\left(\sqrt{2x-x^2}-x\right)dx=\frac{\pi}{2}-1$

20. Let $\vec{a} = \hat{i} + 2\hat{j} - 3k$ and $\vec{b} = 2\hat{i} - 3\hat{j} + 5k$. If $\vec{r} \times \vec{a} = \vec{b} \times \vec{r}$, $\vec{r}.\left(\alpha\hat{i} + 2\hat{j} + k\right) = 3$ and $\vec{r}.\left(2\hat{i} + 5\hat{j} - \alpha k\right) = -1$, $\alpha \in R$, then the value of $\alpha + \left|\vec{r}\right|^2$ is equal to:

(a) 9 (b) 15 (c) 13 (d) 11

Ans. (b)

Solution:

$\vec{r} \times \vec{a} = \vec{b} \times \vec{r} \Rightarrow \vec{r} \times \left(\vec{a} + \vec{b}\right) = 0$

$\vec{r} = \vec{\lambda}\left(\vec{a} + \vec{b}\right) \Rightarrow \vec{r} = \vec{\lambda}\left(\hat{i} + 2\hat{j} - 3k + 2\hat{i} - 3\hat{j} + 5k\right)$

$\vec{r} = \vec{\lambda}\left(3\hat{i} - \hat{j} + 2k\right)$ ...(1)

$\vec{r}.\left(\alpha\hat{i} + 2\hat{j} + k\right) = 3$

Put $\vec{r}$ from (1) $\alpha\lambda = 1$ ...(2)

$\vec{r}.\left(2\hat{i} + 5\hat{j} - \alpha k\right) = -1$

Put $\vec{r}$ from (1) $2\lambda\alpha - \lambda = 1$ ...(3)

Solve (2) & (3)

$\alpha = 1, \lambda = 1$

$\Rightarrow \vec{r} = 3\hat{i} - \hat{j} + 2k$

$\left|\vec{r}\right|^2 = 14$ & $\alpha = 1$

$\alpha + \left|\vec{r}\right|^2 = 15$

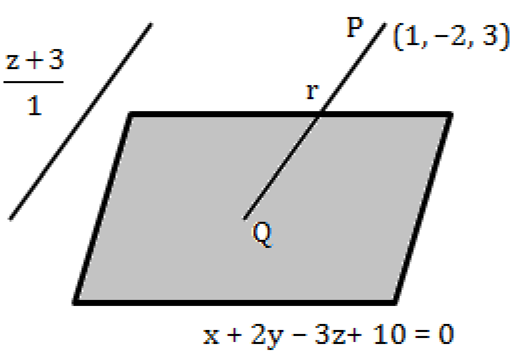
Integer Type

1. If the distance of the point (1, –2, 3) from the plane x + 2y – 3z + 10 = 0 measured parallel to the line, $\frac{x-1}{3} = \frac{2-y}{m} = \frac{z+3}{1}$ is $\sqrt{\frac{7}{2}}$, then the value of $|m|$ is equal to _______.

Ans. 2

Solution:

$\frac{x-1}{3} = \frac{y-2}{-m} = \frac{z+3}{1}$



DC of line $\equiv \left(\frac{3}{\sqrt{m^2+10}}, \frac{-m}{\sqrt{m^2+10}}, \frac{1}{\sqrt{m^2+10}}\right)$

$$Q \equiv \left(1+\frac{3r}{\sqrt{m^2+10}}, -2+\frac{-mr}{\sqrt{m^2+10}}, 3+\frac{r}{\sqrt{m^2+10}}\right)$$

Q lies on x + 2y – 3z + 10 = 0

$$1+\frac{3r}{\sqrt{m^2+10}}-4-\frac{2mr}{\sqrt{m^2+10}}-9-\frac{3r}{\sqrt{m^2+10}}+10=0$$

$$\Rightarrow \frac{r}{\sqrt{m^2+10}}(3-2m-3)=2$$

$$\Rightarrow \frac{r}{\sqrt{m^2+10}}(-2m)=2$$

2. Consider the statistics of two sets of observations as follows:

	Size	Mean	Variance
Observation I	10	2	2
Observation II	n	3	1

If the variance of the combined set of these two observations is $\frac{17}{9}$, then the value of n is equal to _________ .

Ans. 5

Solution:

$$\sigma^2=\frac{n_1\sigma_1^2+n_2\sigma_2^2}{n_1+n_2}+\frac{n_1n_2}{(n_1+n_2)}\left(\bar{x}_1-\bar{x}_2\right)^2$$

$$n_1=10, n_2=n, \sigma_1^2=2, \sigma_2^2=1$$

$$\bar{x}_1=2, \bar{x}_2=3, \sigma^2=\frac{17}{9}$$

$$\frac{17}{9}=\frac{10\times 2+n}{n+10}+\frac{10n}{(n+10)^2}(3-2)^2$$

$$\Rightarrow \frac{17}{9}=\frac{(n+20)(n+10)+10n}{(n+10)^2}$$

$$\Rightarrow 17n^2+1700+340n=90n+9(n^2+30n+200)$$

$$\Rightarrow 8n^2-20n-100=0$$

$$2n^2-5n-25=0$$

$$\Rightarrow (2n+5)(n-5)=0 \Rightarrow n=\frac{-5}{2}, 5$$

↓ (Rejected)

Hence n = 5

3. Let $A=\begin{bmatrix}a_1\\a_2\end{bmatrix}$ and $B=\begin{bmatrix}b_1\\b_2\end{bmatrix}$ be two 2×1 matrices with real entries such that A = XB, where $X=\frac{1}{\sqrt{3}}\begin{bmatrix}1 & -1\\1 & k\end{bmatrix}$, and $k\in R$. If $a_1^2+a_2^2=\frac{2}{3}\left(b_1^2+b_2^2\right)$ and $\left(k^2+1\right)b_2^2\neq -2b_1b_2$, then the value of k is _____ .

Ans. 1

Solution:

A = XB

$$\begin{bmatrix}a_1\\a_2\end{bmatrix}=\frac{1}{\sqrt{3}}\begin{bmatrix}1 & -1\\1 & k\end{bmatrix}\begin{bmatrix}b_1\\b_2\end{bmatrix}$$

$$\begin{bmatrix}\sqrt{3}a_1\\\sqrt{3}a_2\end{bmatrix}=\begin{bmatrix}b_1-b_2\\b_1+kb_2\end{bmatrix}$$

$b_1-b_2=\sqrt{3}\,a_1$ ...(1)

$b_1+kb_2=\sqrt{3}\,a_2$ ...(2)

Given, $a_1^2+a_2^2=\frac{2}{3}\left(b_1^2+b_2^2\right)$

$(1)^2+(2)^2$

$(b_1+b_2)^2+(b_1+kb_2)^2=3\left(a_1^2+a_2^2\right)$

$$a_1^2+a_2^2=\frac{2}{3}b_1^2+\frac{\left(1+k^2\right)}{3}b_2^2+\frac{2}{3}b_1b_2\left(k-1\right)$$

Given, $a_1^2+a_2^2=\frac{2}{3}b_1^2+\frac{2}{3}b_2^2$

On comparing we get

$\frac{k^2+1}{3}=\frac{2}{3}\Rightarrow k^2+1=2$

$\Rightarrow k=\pm 1$ ...(3)

& $\frac{2}{3}\left(k-1\right)=0\ \Rightarrow k=1$ ...(4)

From both we get k = 1

4. For real numbers α,β,γ and δ, if $\int\frac{\left(x^2-1\right)+\tan^{-1}\left(\frac{x^2+1}{x}\right)}{\left(x^4+3x^2+1\right)\tan^{-1}\left(\frac{x^2+1}{x}\right)}dx$

$$=\alpha\log_e\left(\tan^{-1}\left(\frac{x^2+1}{x}\right)\right)+\beta\tan^{-1}\left(\frac{\gamma\left(x^2-1\right)}{x}\right)+\delta\tan^{-1}\left(\frac{x^2+1}{x+C}\right)$$

where C is an arbitrary constant, then the value of $10\left(\alpha+\beta\gamma+\delta\right)$ is equal to ________ .

Ans. 6

Solution:

$$\int\frac{\left(x^2-1\right)dx}{\left(x^4+3x^2+1\right)\tan^{-1}\left(x+\frac{1}{x}\right)}+\int\frac{dx}{x^4+3x^2+1}$$

$$\int \frac{\left(1-\frac{1}{x^2}\right)}{\left(\left(x+\frac{1}{x}\right)^2+1\right)\tan^{-1}\left(x+\frac{1}{x}\right)}+\frac{1}{2}\int\frac{(x^2+1)-(x^2-1)dx}{x^4+3x^2+1}$$

Put $\tan^{-1}\left(x+\frac{1}{x}\right)=t$

$$\int\frac{dt}{t}+\frac{1}{2}\int\frac{\left(1+\frac{1}{x^2}\right)dx}{\left(x-\frac{1}{x}\right)^2+5}-\frac{1}{2}\int\frac{\left(1-\frac{1}{x^2}\right)dx}{\left(x+\frac{1}{x}\right)^2+1}$$

Put $x-\frac{1}{x}=y,\ x+\frac{1}{x}=z$

$$\log_e t+\frac{1}{2}\int\frac{dy}{y^2+5}-\frac{1}{2}\int\frac{dz}{z^2+1}$$

$$=\log_e\tan^{-1}\left(x+\frac{1}{x}\right)+\frac{1}{2\sqrt{5}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{5}x}\right)-\frac{1}{2}\tan^{-1}\left(\frac{x^2+1}{x}\right)+C$$

$\alpha=1,\ \beta=\frac{1}{2\sqrt{5}},\ \gamma=\frac{1}{\sqrt{5}},\ \delta=\frac{-1}{2}$

or

$\alpha=1,\beta=\frac{-1}{2\sqrt{5}},\gamma=\frac{-1}{\sqrt{5}},\delta=\frac{-1}{2}$

$10(\alpha+\beta\gamma+\delta)=10\left(1+\frac{1}{10}-\frac{1}{2}\right)=6$

5. Let f: $R\to R$ and $g:R\to R$ be defined as $f(x)=\begin{cases}x+a, & x<0\\ |x-1|, & x\ge 0\end{cases}$ and $g(x)=\begin{cases}x+1, & x<0\\ (x-1)^2+b, & x\ge 0\end{cases}$

where a, b are non-negative real numbers. If (gof) (x) is continuous for all $x\in R$, then a + b is equal to ________ .

Ans. 1

Solution:

$$g[f(x)]=\begin{cases}f(x)+1 & f(x)<0\\ (f(x)-1)^2+b & f(x)\ge 0\end{cases}$$

$$g[f(x)]=\begin{cases}x+a+1 & x+a<0\ \&\ x<0\\ |x-1|+1 & |x-1|<0\ \&\ x\ge 0\\ (x+a-1)^2+b & x+a\ge 0\ \&\ x<0\\ (|x-1|-1)^2+b & |x-1|\ge 0\ \&\ x\ge 0\end{cases}$$

$$g[f(x)] = \begin{cases} x+a+1 & x\in(-\infty,-a)\ \&\ x\in(-\infty,0) \\ |x-1|+1 & x\in\phi \\ (x+a-1)^2+b & x\in[-a,\infty)\ \&\ x\in(-\infty,0) \\ (|x-1|-1)^2+b & x\in R\ \&\ x\in[0,\infty) \end{cases}$$

$$g[f(x)] = \begin{cases} x+a+1 & x\in(-\infty,-a) \\ (x+a-1)^2+b & x\in[-a,0) \\ (|x-1|-1)^2+b & x\in[0,\infty) \end{cases}$$

g(f(x)) is continuous

at x = – a & at x = 0

1 = b + 1 & (a – 1)2 + b = b

b = 0 & a = 1

$\Rightarrow$ a + b = 1

6. Let $\frac{1}{16}$, a and b be in G.P. and $\frac{1}{a}, \frac{1}{b}, 6$ be in A.P., where a, b > 0. Then 72(a + b) is equal to ________ .

Ans. 14

Solution:

$$a^2 = \frac{b}{16} \Rightarrow \frac{1}{b} = \frac{1}{16a^2}$$

$$\frac{2}{b} = \frac{1}{a} + 6$$

$$\frac{1}{8a^2} = \frac{1}{a} + 6$$

$$\frac{1}{a^2} - \frac{8}{a} - 48 = 0$$

$$\frac{1}{a} = 12, -4 \Rightarrow a = \frac{1}{12}, -\frac{1}{4}$$

$$a = \frac{1}{12}, a > 0$$

$$b = 16a^2 = \frac{1}{9}$$

$$\Rightarrow 72(a+b) = 6 + 8 = 14$$

7. In ΔABC, the lengths of sides AC and AB are 12cm and 5cm, respectively. If the area of ΔABC is 30 cm² and R and r are respectively the radii of circumcircle and incircle of ΔABC, then the value of 2R + r (in cm) is equal to ___________ .

Ans. 15

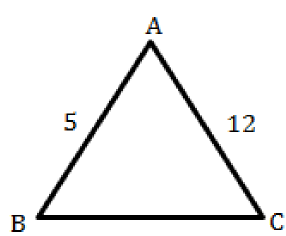
Solution:

$$\Delta = \frac{1}{2}.5.12.\sin A = 30$$

sin A = 1

$A = 90^0 \Rightarrow BC = 13$

BC = 2R = 13



$r = \frac{\Delta}{S} = \frac{30}{15} = 2$

2R + r = 15

8. Let n be a positive integer. Let $A = \sum_{k=0}^{n}(-1)^k\, {}^nC_k\left[\left(\frac{1}{2}\right)^k + \left(\frac{3}{4}\right)^k + \left(\frac{7}{8}\right)^k + \left(\frac{15}{16}\right)^k + \left(\frac{31}{32}\right)^k\right]$

If $63A = 1 - \frac{1}{2^{30}}$, then n is equal to __________ .

Ans. 6

Solution:

$$A = \sum_{k=0}^{n} {}^nC_k\left[\left(-\frac{1}{2}\right)^k + \left(\frac{-3}{4}\right)^k + \left(\frac{7}{8}\right)^k + \left(\frac{15}{16}\right)^k + \left(\frac{31}{32}\right)^k\right]$$

$$A = \left(1-\frac{1}{2}\right)^n + \left(1-\frac{3}{4}\right)^n + \left(1-\frac{7}{8}\right)^n + \left(1-\frac{15}{16}\right)^n + \left(1-\frac{31}{32}\right)^n$$

$$A = \frac{1}{2^n} + \frac{1}{4^n} + \frac{1}{8^n} + \frac{1}{16^n} + \frac{1}{32^n}$$

$$A = \frac{1}{2^n}\left(\frac{1-\left(\frac{1}{2^n}\right)^5}{1-\frac{1}{2^n}}\right) \Rightarrow A = \frac{\left(1-\frac{1}{2^{5n}}\right)}{\left(2^n-1\right)}$$

$\left(2^n-1\right)A = 1 - \frac{1}{2^{5n}}$, Given $63A = I - \frac{1}{2^{30}}$

Clearly 5n = 30

n = 6

9. Let $\vec{c}$ be a vector perpendicular to the vectors $\vec{a} = \hat{i} + \hat{j} - k$ and $\vec{b} = \hat{i} + 2\hat{j} + k$.

If $\vec{c}.\left(\hat{i} + \hat{j} + 3k\right) = 8$ then the value of $\vec{c}.\left(\vec{a} \times \vec{b}\right)$ is equal to _________ .

Ans. 28

Solution:

$\vec{c} = \lambda\left(\vec{a} \times \vec{b}\right)$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & k \\ 1 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix}$$

$\left(\vec{a} \times \vec{b}\right) = 3\hat{i} - 2\hat{j} + k$

$\vec{c}.\left(\hat{i} + \hat{j} + 3k\right) = \lambda\left(3\hat{i} - 2\hat{j} + k\right).\left(\hat{i} + \hat{j} + 3k\right)$

$\Rightarrow \lambda(4) = 8 \Rightarrow \lambda = 2$

$\vec{c} = 2\left(\vec{a} \times \vec{b}\right)$

$\vec{c}.\left(\vec{a} \times \vec{b}\right) = 2\left|\vec{a} \times \vec{b}\right|^2 = 28$

10. Let $S_n(x) = \log_{a^{1/2}} x + \log_{a^{1/3}} x + \log_{a^{1/6}} x + \log_{a^{1/11}} x + \log_{a^{1/18}} x + \log_{a^{1/27}} x + ...$ up to n-terms, where a > 1. If $S_{24}(x) = 1093$ and $S_{12}(2x) = 265$, then value of a is equal to ________ .

Ans. 16

Solution:

$S_n(x) = (2 + 3 + 6 + 11 + 18 + 27 + ... + \text{n-terms}) \log_a x$

Let $S_1 = 2 + 3 + 6 + 11 + 18 + 27 + ... + T_n$

$S_1 = 2 + 3 + 6 + ... + T_n$

$T_n = 2 + 1 + 3 + 5 + ... + \text{n terms}$

$T_n = 2 + (n-1)^2$

$$S_1 = \sum T_n = 2n + \frac{(n-1)n(2n-1)}{6}$$

$$\Rightarrow S_n(x) = \left(2n + \frac{n(n-1)(2n-1)}{6}\right)\log_a x$$

$S_{24}(x) = 1093$ (Given)

$$\log_a x\left(48 + \frac{23.24.47}{6}\right) = 1093$$

$$\log_a x = \frac{1}{4} \quad ....(1)$$

$S_{12}(2x) = 265$

$S_{12}(2x) = 265$

$$\log_a(2x)\left(24 + \frac{11.12.23}{6}\right) = 265$$

$$\log_a 2x = \frac{1}{2} \quad ....(2)$$

(2) – (1)

$$\log_a 2x - \log_a x = \frac{1}{4}$$

$$\log_a 2 = \frac{1}{4} \Rightarrow a = 16$$