JEE-Main Mathematics Solution 16.3.2021 Shift-1

1. The number of elements in the set

$\{x \in R : (|x| - 3)|x + 4| = 6\}$ is equal to

(a) 3 (b) 2 (c) 4 (d) 1

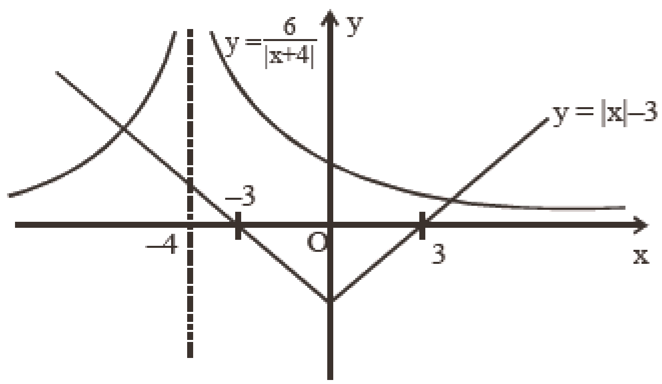
Ans. (b)

Solution:

$x \neq -4$

$(|x| - 3)(|x| + 4) = 6$

$\Rightarrow |x| - 3 = \frac{6}{|x+4|}$



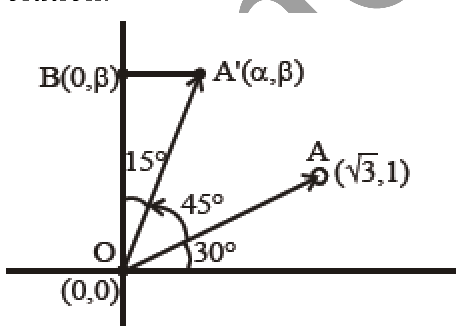
No. of solutions = 2

2. Let a vector $\alpha\hat{i} + \beta\hat{j}$ be obtained by rotating the vector $\sqrt{3}\hat{i} + \hat{j}$ by an angle 45^0 about the origin in counter-wise direction in the first quadrant. Then the area of triangle having vertices $(\alpha, \beta), (0, \beta)$ and $(0, 0)$ is equal to

(a) $\frac{1}{2}$ (b) 1 (c) $\frac{1}{\sqrt{2}}$ (d) $2\sqrt{2}$

Ans. (a)

Solution:



Area of $\Delta(OA'B) = \frac{1}{2} OA' \cos 15^0 \times OA' \sin 15^0$

$= \frac{1}{2}(OA')^2 \frac{\sin 30^0}{2}$

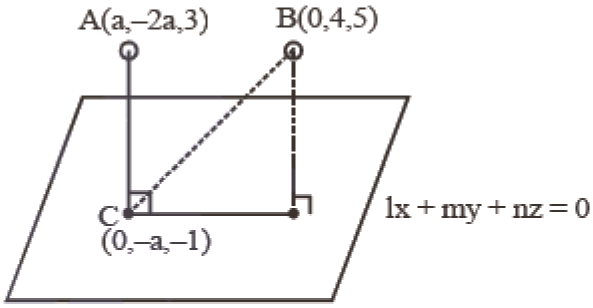
$= (3+1) \times \frac{1}{8} = \frac{1}{2}$

3. If for a > 0, the feet of perpendiculars from the points A(a, – 2a, 3) and B(0, 4, 5) on the plane lx + my + nz = 0 are points C(0, – a, – 1) and D respectively, then the length of line segment CD is equal to:

(a) $\sqrt{31}$ (b) $\sqrt{41}$ (c) $\sqrt{55}$ (d) $\sqrt{66}$

Ans. (d)

Solution:



C lies on plane $\Rightarrow -ma-n=0 \Rightarrow \frac{m}{n}=-\frac{1}{a}$....(1)

$\overrightarrow{CA} || l\hat{i}+m\hat{j}+n\hat{k}$

$\frac{a-0}{l}=\frac{-a}{m}=\frac{4}{m} \Rightarrow \frac{m}{n}=-\frac{a}{4}$ ..(2)

From (1) & (2)

$-\frac{1}{a}=\frac{-a}{4} \Rightarrow a^2=4 \Rightarrow a=2$ (since a > 0)

From (2) $\frac{m}{n}=\frac{-1}{2}$

Let m = – t $\Rightarrow$ $n=2t$

$\frac{2}{l}=\frac{-2}{-t} \Rightarrow l=t$

So plane: t(x – y +2z) = 0

BD = $\frac{6}{\sqrt{6}}=\sqrt{6}$ $C \equiv (0,-2,-1)$

$CD=\sqrt{BC^2-BD^2}$

$=\sqrt{\left(0^0+6^2+6^2\right)-\left(\sqrt{6}\right)^2}$

$=\sqrt{66}$

4. Consider three observation a, b and c such that b = a + c. If the standard deviation of a + 2, b + 2, c + 2 is d, then which of the following is true?

(a) $b^2 = 3(a^2 + c^2) + 9d^2$

(b) $b^2 = a^2 + c^2 + 3d^2$

(c) $b^2 = 3(a^2 + c^2 + d^2)$

(d) $b^2 = 3(a^2 + c^2) - 9d^2$

Ans. (d)

Solution:

For a, b, c

mean = $\frac{a+b+c}{3}\left(=\bar{x}\right)$

b = a + c

$\Rightarrow \bar{x}=\frac{2b}{3}$ ...(1)

S. D. (a + 2, b + 2, c + 2) = S. D. (a, b, c) = d

$\Rightarrow d^2=\frac{a^2+b^2+c^2}{3}-\left(\bar{x}\right)^2$

$\Rightarrow d^2=\frac{a^2+b^2+c^2}{3}-\frac{4b^2}{9}$

$\Rightarrow 9d^2 = 3(a^2 + b^2 + c^2) - 4b^2$

$\Rightarrow b^2=3\left(a^2+c^2\right)-9d^2$

5. If for $x\in\left(0,\frac{\pi}{2}\right)$, $\log_{10}\sin x+\log_{10}\cos x=-1$ and $\log_{10}(\sin x + \cos x) = \frac{1}{2}(\log_{10}n - 1)$, n > 0, then the value of n is equal to:

(a) 20 (b) 12 (c) 9 (d) 16

Ans. (b)

Solution:

$x\in\left(0,\frac{\pi}{2}\right)$

$\log_{10}\sin x + \log_{10}\cos x = -1$

$\Rightarrow \log_{10}\sin x.\cos x=-1$

$\Rightarrow \sin x.\cos x=\frac{1}{10}$ ...(1)

$\log_{10}\left(\sin x+\cos x\right)=\frac{1}{2}\left(\log_{10}n-1\right)$

$\Rightarrow \sin x+\cos x=10^{\left(\log_{10}\sqrt{n}-\frac{1}{2}\right)}=\sqrt{\frac{n}{10}}$

by squaring

$1+2\sin x.\cos x=\frac{n}{10}$

$\Rightarrow 1+\frac{1}{5}=\frac{n}{10} \Rightarrow n=12$

6. Let $A=\begin{bmatrix} i & -i \\ -i & i \end{bmatrix}$, $i=\sqrt{-1}$. Then, the system of linear equations $A^8\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 8 \\ 64 \end{bmatrix}$ has:

(a) A unique solution

(b) Infinitely many solutions

(c) No solution

(d) Exactly two solutions

Ans. (c)

Solution:

$$A=\begin{bmatrix} i & -i \\ -i & i \end{bmatrix}$$

$$A^2=\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}=2\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$A^4=2^2\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}=8\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$A^8=64\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}=128\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$A^8\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 8 \\ 64 \end{bmatrix}$$

$$\Rightarrow\ 128\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 8 \\ 64 \end{bmatrix}$$

$$\Rightarrow\ 128\begin{bmatrix} x-y \\ -x+y \end{bmatrix}=\begin{bmatrix} 8 \\ 64 \end{bmatrix}$$

$$\Rightarrow\ x-y=\frac{1}{16} \qquad ...(1)$$

$$\&\ -x+y=\frac{1}{2} \qquad ...(2)$$

$\Rightarrow$ From (1) & (2) : No solution.

7. If the three normals drawn to the parabola, $y^2 = 2x$ pass through the point $(a, 0)$ $a \neq 0$, then 'a' must be greater than:

(a) $\frac{1}{2}$ (b) $-\frac{1}{2}$ (c) – 1 (d) 1

Ans. (d)

Solution:

For standard parabola

For more than 3 normals (on axis)

$x > \frac{L}{2}$ (where L is length of L. R.)

For $y^2 = 2x$

L. R. = 2

for (a, 0)

$a > \frac{L.R.}{2} \Rightarrow a > 1$

8. Let the position vectors of two points P and Q be $3\hat{i}-\hat{j}+2k$ and $\hat{i}+2\hat{j}-4k$, respectively. Let R and S be two points such that the direction ratios of lines PR and QS are (4, –1, 2) and (–2, 1, – 2), respectively. Let lines PR and QS intersect at T. If the vector $\overrightarrow{TA}$ is perpendicular to both $\overrightarrow{PR}$ and $\overrightarrow{QS}$ and the length of vector $\overrightarrow{TA}$ is $\sqrt{5}$ units, then the modulus of a position vector of A is:

(a) $\sqrt{482}$ (b) $\sqrt{171}$ (c) $\sqrt{5}$ (d) $\sqrt{227}$

Ans. (b)

Solution:

P(3, –1, 2)

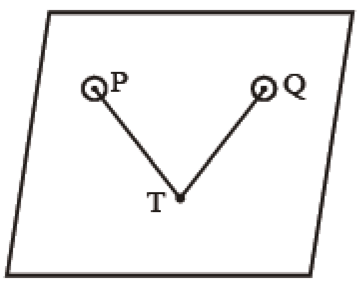
Q(1, 2, –4)

$\overrightarrow{PR} \,||\, 4\hat{i}-\hat{j}+2k$

$\overrightarrow{QS} \,||\, -2\hat{i}+\hat{j}-2k$

dr's of normal to the plane containing P, T & Q will be proportional to:

$$\begin{vmatrix} \hat{i} & \hat{j} & k \\ 4 & -1 & 2 \\ -2 & 1 & -2 \end{vmatrix}$$



$$\therefore \frac{\ell}{0}=\frac{m}{4}=\frac{n}{2}$$

For point, T: $\overrightarrow{PT}=\frac{x-3}{4}=\frac{y+1}{-1}=\frac{z-2}{2}=\lambda$

$\overrightarrow{QT}=\frac{x-1}{-2}=\frac{y-1}{1}=\frac{z+4}{-2}=\mu$

$T:(4\lambda+3,-\lambda-1,2\lambda+2)$

$\cong(2\mu+1,\mu+2,-2\mu-4)$

$4\lambda+3=-2\mu+1 \Rightarrow \quad 2\lambda+\mu=-1$

$\lambda+\mu=-3 \quad \Rightarrow \quad \lambda=2$

$\&\ \mu=-5 \quad\quad \lambda+\mu=-3 \Rightarrow \lambda=2$

So point T: (11, –3, 6)

$$\overrightarrow{OA}=\left(11\hat{i}-3\hat{j}+6k\right)\pm\left(\frac{2\hat{j}+k}{\sqrt{5}}\right)\sqrt{5}$$

$\overrightarrow{OA} = \left(11\hat{i} - 3\hat{j} + 6k\right) \pm \left(2\hat{j} + k\right)$

$\overrightarrow{OA} = 11\hat{i} - \hat{j} + 7k$

or

$9\hat{i} - 5\hat{j} + 5k$

$\left|\overrightarrow{OA}\right| = \sqrt{121 + 1 + 49} = \sqrt{171}$

or

$\sqrt{81 + 25 + 25} = \sqrt{131}$

9. Let the functions f: $R \to R$ and g: $R \to R$ be defined as:

$f(x) = \begin{cases} x + 2, x < 0 \\ x^2, x \geq 0 \end{cases}$ and $g(x) = \begin{cases} x^3, \ x < 1 \\ 3x - 2, x \geq 1 \end{cases}$

Then, the number of points in R where (fog) (x) is NOT differentiable is equal to:

(a) 3 (b) 1 (c) 0 (d) 2

Ans. (b)

Solution:

$$f(g(x)) = \begin{cases} g(x) + 2, g(x) < 0 \\ (g(x))^2, g(x) \geq 0 \end{cases}$$

$$= \begin{cases} x^3 + 2 & , \ x < 0 \\ x^6 & , \ x \in [0, 1) \\ (3x - 2)^2 & , \ x \in [1, \infty) \end{cases}$$

$$(fog(x))' = \begin{cases} 3x^2, & x < 0 \\ 6x^5, & x \in (0, 1) \\ 2(3x - 2) \times 3, & x \in (1, \infty) \end{cases}$$

At '0'

L. H. L. $\neq$ R. H. L. (Discontinuous)

At '1'

L. H. D. = 6 = R. H. D.

$\Rightarrow$ fog (x) is differentiable for $x \in R - \{0\}$

10. Which of the following Boolean expression is a tautology?

(a) $(p \wedge q) \vee (p \vee q)$

(b) $(p \wedge q)$ v $(p \to q)$

(c) $(p \wedge q) \wedge (p \to q)$

(d) $(p \wedge q) \to (p \to q)$

Ans. (d)

Solution:

P	Q	$p \wedge q$	$p \to q$	$(p \wedge q) \to (p \to q)$
T	T	T	T	T
T	F	F	F	T
F	T	F	T	T
F	F	F	T	T

$(p \wedge q) \to (p \to q)$ is tautology

11. Let a complex number z, $|z| \neq 1$, satisfy $\log_{\frac{1}{\sqrt{1}}}\left(\frac{|z|+11}{(|z|-1)^2}\right) \leq 2.$

Then, the largest value of $|z|$ is equal to____ .

(a) 8 (b) 7 (c) 6 (d) 5

Ans. (b)

Solution:

$$\log_{\frac{1}{\sqrt{2}}}\left(\frac{|z|+11}{(|z|-1)^2}\right) \leq 2$$

$$\frac{|z|+11}{(|z|-1)^2} \geq \frac{1}{2}$$

$$2|z|+22 \geq (|z|-1)^2$$

$$2|z|+22 \geq |z|^2+1-2|z|$$

$$|z|^2-4|z|-21 \leq 0$$

$$\Rightarrow |z| \leq 7$$

$\therefore$ Largest value of $|z|$ is 7

12. If n is the number of irrational terms in the expansion of $(3^{1/4} + 5^{1/8})^{60}$, then $(n - 1)$ is divisible by:

(a) 26 (b) 30 (c) 8 (d) 7

Ans. (a)

Solution:

$(3^{1/4} + 5^{1/8})^{60}$

$$^{60}C_r\left(3^{1/4}\right)^{60-r}.\left(5^{1/8}\right)^r$$

$$^{60}C_r(3)^{\frac{60-r}{4}}.5^{\frac{r}{8}}$$

For rational terms.

$$\frac{r}{8}=k; \quad 0 \leq r \leq 60$$

$$0 \leq 8k \leq 60$$

$$0 \leq k \leq \frac{60}{8}$$

$0 \le k \le 7.5$

k = 0, 1, 2, 3, 4, 5, 6, 7

$\frac{60-8k}{4}$ is always divisible by 4 for all value of k.

Total rational terms = 8

Total terms = 61

irrational terms = 53

n – 1= 53 – 1 = 52

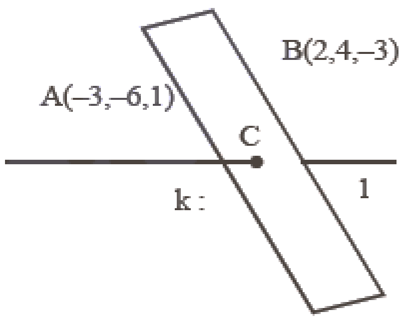
52 is divisible by 26.

13. Let P be a plane lx + my + nz = 0 containing the line, $\frac{1-x}{1}=\frac{y+4}{2}=\frac{z+2}{3}$. If plane p divides the line segment AB joining points A(–3, –6, 1) and B(2, 4, – 3) in ratio k : 1 then the value of k is equal to:

(a) 1. 5 (b) 3 (c) 2 (d) 4

Ans. (c)

Solution:



Point C is

$\left(\frac{2k-3}{k+1},\frac{4k-6}{k+1},\frac{-3k+1}{k+1}\right)$

$\frac{x-1}{-1}=\frac{y+4}{2}=\frac{z+2}{3}$

Plane lx + my + nz = 0

l(–1) + m(2) + n(3) = 0

–l + 2m + 3n = 0 ...(1)

It also satisfy point (1, –4, – 2)

l – 4m – 2n = 0 ...(2)

Solving (1) and (2)

2m + 3n = 4m + 2n

n = 2m

l – 4m – 4m = 0

l = 8m

$\frac{l}{8}=\frac{m}{1}=\frac{n}{2}$

l : m : n = 8 : 1 : 2

Plane is 8x + y + 2z = 0

It will satisfy point C

$$8\left(\frac{2k-3}{k+1}\right)+\left(\frac{4k-6}{k+1}\right)+2\left(\frac{-3k+1}{k+1}\right)=0$$

$16k - 24 + 4k - 6 - 6k + 2 = 0$

$14k = 28 \quad \therefore \quad k=2$

14. The range of $a \in R$ for which the function $f(x) = (4a - 3)(x + \log_e 5) + 2(a - 7)\cot\left(\frac{x}{2}\right)\sin^2\left(\frac{x}{2}\right)$, $x \neq 2n\pi, n \in N$, has critical points, is:

(a) $(-3, 1)$ (b) $\left[-\frac{4}{3}, 2\right]$ (c) $[1, \infty)$ (d) $(-\infty, -1]$

Ans. (b)

Solution:

$f(x) = (4a - 3)(x + \log_e 5) + (a - 7)\sin x$

$f(x) = (4a - 3)(1) + (a - 7)\cos x = 0$

$\Rightarrow \cos x = \frac{3-4a}{a-7}$

$-1 \leq \frac{3-4a}{a-7} < 1$

$\frac{3-4a}{a-7}+1 \geq 0$

$\frac{3-4a+a-7}{a-7} \geq 0$

$\frac{-3-4}{a-7} \geq 0$

$\frac{3a+4}{a-7} \leq 0$

$\frac{3-4a}{a-7} < 1$

$\frac{3-4a}{a-7} - 1 < 0$

$\frac{3-4a-a+7}{a-7} < 0$

$\frac{-5a+10}{a-7} < 0$

$\frac{5a-10}{a-7} > 0$

$\frac{5(a-2)}{a-7} > 0$

$\alpha \in \left[-\frac{4}{3}, 2\right)$

Check end point $\left[-\frac{4}{3}, 2\right)$

15. A pack of cards has one card missing. Two cards are drawn randomly and are found to be spades. The probability that the missing card is not a spade, is:

(a) $\frac{3}{4}$ (b) $\frac{52}{867}$ (c) $\frac{39}{50}$ (d) $\frac{22}{425}$

Ans. (c)

Solution:

E_1: Event denotes spade is missing

$$P(E_1)=\frac{1}{4};\ P(\bar{E}_1)=\frac{3}{4}$$

A: Event drawn two cards are spade

$$P(A)=\frac{\frac{1}{4}\times\left(\frac{^{12}C_2}{^{51}C_2}\right)+\frac{3}{4}\times\left(\frac{^{13}C_2}{^{51}C_2}\right)+\frac{3}{4}\times\left(\frac{^{13}C_2}{^{51}C_2}\right)}{\frac{1}{4}\times\left(\frac{^{12}C_2}{^{51}C_2}\right)+\frac{3}{4}\times\left(\frac{^{13}C_2}{^{51}C_2}\right)}=\frac{39}{50}$$

16. Let [x] denote greatest integer less than or equal to x. If for

$n\in N,\ (1-x+x^3)^n=\sum_{j=0}^{3n}a_jx^j$, then $\sum_{j=0}^{\left[\frac{3n}{2}\right]}a_{2j}+4\sum_{j=0}^{\left[\frac{3n-1}{2}\right]}a_{2j}+1$ is equal to:

(a) 2 (b) 2^{n-1} (c) 1 (d) n

Ans. (c)

Solution:

$$(1-x+x^3)^n=\sum_{j=0}^{3n}a_jx^j$$

$$(1-x+x^3)^n=a_0+a_1x+a_2x^2.....+a_{3n}x^{3n}$$

$\sum_{j=0}^{\left[\frac{3n}{2}\right]}a_{2j}$ = Sum of $a_0+a_2+a_4$ …….

$\sum_{j=0}^{\left[\frac{3n-1}{2}\right]}a_{2j}+1$ = Sum of $a_1+a_3+a_5$ ……

put x = 1

$1=a_0+a_1+a_2+a_3\ ......+a_{3n}$ ….(A)

Put x = – 1

$1=a_0-a_1+a_2-a_3\ .....+(-1)^{3n}a_{3n}$ …(B)

Solving (A) and (B)

$a_0+a_2+a_4\ ....=1$

$a_1+a_3+a_5\ .....=0$

$$\sum_{j=0}^{\left[\frac{3n}{2}\right]}a_{2j}+4\sum_{j=0}^{\left[\frac{3n-1}{2}\right]}a_{2j+1}=1$$

17. If y = y(x) is the solution of the differential equation, $\frac{dy}{dx}+2y\tan x=\sin x,\ y\left(\frac{\pi}{3}\right)=0$, then the maximum value of the function y(x) over R is equal to:

(a) 8 (b) $\frac{1}{2}$ (c) $-\frac{15}{4}$ (d) $\frac{1}{8}$

Ans. (d)

Solution:

$$\frac{dy}{dx}+2y\tan x=\sin x$$

I.F. = $e^{\int 2\tan x\,dx}=e^{2\ell n\,\sec x}$

I. F. = $\sec^2 x$

$$y.\left(\sec^2 x\right)=\int \sin x.\sec^2 x\,dx$$

$$y.\left(\sec^2 x\right)=\int \sec x\tan x\,dx$$

$$y.\left(\sec^2 x\right)=\sec x+C$$

$$x=\frac{\pi}{3};\ y=0$$

$$\Rightarrow\ C=-2$$

$$\Rightarrow\ y=\frac{\sec x-2}{\sec^2 x}=\cos x-2\cos^2 x$$

$$y=t-2t^2\Rightarrow\frac{dy}{dt}=1-4t=0\ \Rightarrow t=\frac{1}{4}$$

$$\therefore\ \max=\frac{1}{4}-\frac{1}{8}=\frac{2-1}{8}=\frac{1}{8}$$

18. The locus of the mid-points of the chord of the circle, $x^2 + y^2 = 25$ which is tangent to the hyperbola, $\frac{x^2}{9}-\frac{y^2}{16}=1$ is:

(a) $(x^2 + y^2)^2 - 16x^2 + 9y^2 = 0$

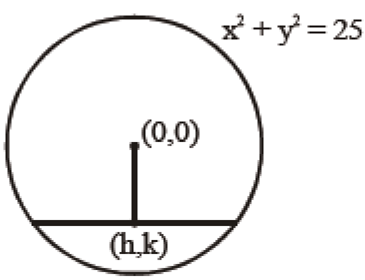
(b) $(x^2 + y^2)^2 - 9x^2 + 144y^2 = 0$

(c) $(x^2 + y^2)^2 - 9x^2 - 16y^2 = 0$

(d) $(x^2 + y^2)^2 - 9x^2 + 16y^2 = 0$

Ans. (d)

Solution:



Equation of chord

$$y-k=-\frac{h}{k}(x-h)$$

$ky - k^2 = - hx + h^2$

$hx + ky = h^2 + k^2$

$$y=-\frac{hx}{k}+\frac{h^2+k^2}{k}$$

tangent to $\frac{x^2}{9}-\frac{y^2}{16}=1$

$c^2 = a^2m^2 - b^2$

$$\left(\frac{h^2+k^2}{k}\right)^2 = 9\left(-\frac{h}{k}\right)^2 - 16$$

$$\left(x^2+y^2\right)^2 = 9x^2 - 16y^2$$

19. The number of roots of the equation, $(81)^{\sin^2 x} = 30$ in the interval $[0, \pi]$ is equal to:

(a) 3 (b) 4 (c) 8 (d) 2

Ans. (b)

Solution:

$$(81)^{\sin^2 x} + (81)^{\cos^2 x} = 30$$

$$(81)^{\sin^2 x} + \frac{(81)^1}{(18)^{\sin^2 x}} = 30$$

$$(81)^{\sin^2 x} = t$$

$$t + \frac{81}{t} = 30$$

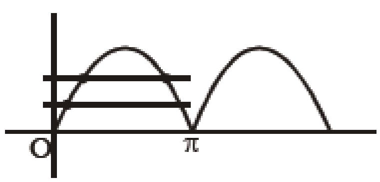
$t^2 - 30t + 81 = 0$

$(t - 3)(t - 27) = 0$

$(81)^{\sin^2 x} = 3^1$ or $(81)^{\sin^2 x} = 3^3$

$3^{4\sin^2 x} = 3^1$ or $3^{4\sin^2 x} = 3^3$

$\sin^2 x = \frac{1}{4}$ or $\sin^2 x = \frac{3}{4}$



Total sol. = 4

20. Let $S_k = \sum_{r=1}^{k} \tan^{-1}\left(\frac{6^r}{2^{2r+1}+3^{2r+1}}\right)$. Then $\lim_{k\to\infty} S_k$ is equal to:

(a) $\tan^{-1}\left(\frac{3}{2}\right)$ (b) $\frac{\pi}{2}$ (c) $\cot^{-1}\left(\frac{3}{2}\right)$ (d) $\tan^{-1}(3)$

Ans. (c)

Solution:

$$S_k = \sum_{r=1}^{k} \tan^{-1}\left(\frac{6^r}{2^{2r+1}+3^{2r+1}}\right)$$

Divide by 3^{2r}

$$\sum_{r=1}^{k} \tan^{-1}\left(\frac{\left(\frac{2}{3}\right)^{r}}{\left(\frac{2}{3}\right)^{2r}.2+3}\right)$$

$$\sum_{r=1}^{k} \tan^{-1}\left(\frac{\left(\frac{2}{3}\right)^{r}}{3\left(\left(\frac{2}{3}\right)^{2r+1}+1\right)}\right)$$

Let $\left(\frac{2}{3}\right)^{r}=1$

$$\sum_{r=1}^{k} \tan^{-1}\left(\frac{\frac{t}{3}}{1+\frac{2}{3}t^{2}}\right)$$

$$\sum_{r=1}^{k} \tan^{-1}\left(\frac{t-\frac{2t}{3}}{1+t.\frac{2t}{3}}\right)$$

$$\sum_{r=1}^{k}\left(\tan^{-1}(t)-\tan^{-1}\left(\frac{2t}{3}\right)\right)$$

$$\sum_{r=1}^{k}\left(\tan^{-1}\left(\frac{2}{3}\right)^{r}-\tan^{-1}\left(\frac{2}{3}\right)^{r+1}\right)$$

$$S_k=\tan^{-1}\left(\frac{2}{3}\right)-\tan^{-1}\left(\frac{2}{3}\right)^{k+1}$$

$$S_\infty=\lim_{k\to\infty}\left(\tan^{-1}\left(\frac{2}{3}\right)-\tan^{-1}\left(\frac{2}{3}\right)^{k+1}\right)$$

$$=\tan^{-1}\left(\frac{2}{3}\right)-\tan^{-1}(0)$$

$$\therefore\ S_\infty=\tan^{-1}\left(\frac{2}{3}\right)=\cot^{-1}\left(\frac{3}{2}\right)$$

Section-B

1. Consider an arithmetic series and a geometric series having four initial terms from the set {11, 8, 21, 16, 26, 32, 4}. If the last terms of these series are the maximum possible four digit numbers, then the number of common terms in these two series is equal to _________.

Ans. 3

Solution:

GP: 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096, 8192

AP: 11, 16, 21, 26, 31, 36

Common terms: 16, 256, 4096 only

2. Let f: $(0,2) \to R$ be defined as $f(x) = \log_2\left(1 + \tan\left(\frac{\pi x}{2}\right)\right)$.

Then, $\lim_{n\to\infty} \frac{2}{n}\left(f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + ... + f(1)\right)$ is equal to ________ .

Ans. 1

Solution:

$$E = 2\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}f\left(\frac{r}{n}\right)$$

$$E = \frac{2}{\ell n2}\int_0^1 \ell n\left(1 + \tan\frac{\pi x}{4}\right)dx \qquad ...(i)$$

replacing $x \to 1 - x$

$$E = \frac{2}{\ell n2}\int_0^1 \ell n\left(1 + \tan\frac{\pi}{4}(1-x)\right)dx$$

$$E = \frac{2}{\ell n2}\int_0^1 \ell n\left(1 + \tan\left(\frac{\pi}{4} - \frac{\pi}{4}x\right)\right)dx$$

$$E = \frac{2}{\ell n2}\int_0^1 \ell n\left(1 + \frac{1 + \tan\frac{\pi}{4}x}{1 + \tan\frac{\pi}{4}x}\right)dx$$

$$E = \frac{2}{\ell n2}\int_0^1 \ell n\left(\frac{2}{1 + \tan\frac{\pi x}{4}}\right)dx$$

$$E = \frac{2}{\ell n2}\int_0^1\left(\ell n2 - \ell n\left(1 + \tan\frac{\pi x}{4}\right)\right)dx \qquad ...(ii)$$

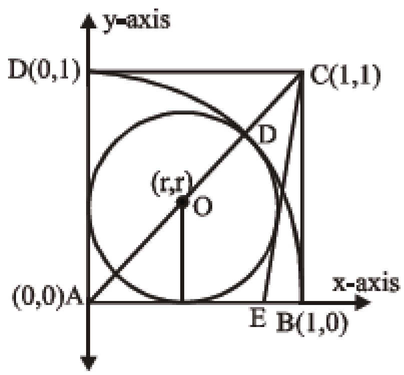
equation (i) + (ii)

E = 1

3. Let ABCD be a square of side of unit length. Let a circle C_1 centered at A with unit radius is drawn. Another circle C_2 which touches C_1 and the lines AD and AB are tangent to it, is also drawn. Let a tangent line from the point C to the circle C_2 meet the side AB at E. If the length EB is $\alpha + \sqrt{3}\beta$, where α, β are integers, then $\alpha + \beta$ is equal to ________ .

Ans. 1

Solution:



Here AO + OD = 1 or $\left(\sqrt{2}+1\right)r=1$

$\Rightarrow\ r=\sqrt{2-1}$

equation of circle $(x-r)^2+(y-r)^2=r^2$

Equation of CE

$y-1=m(x-1)$

$mx-y+1-M=0$

It is tangent to circle

$\therefore\ \left|\frac{mr-r+1-m}{\sqrt{m^2+1}}\right|=r$

$\left|\frac{(m-1)r+1-m}{\sqrt{m^2+1}}\right|=r$

$\frac{(m-1)^2(r-1)^2}{m^2+1}=r^2$

Put $r=\sqrt{2}-1$

On solving $m=2-\sqrt{3},\ 2+\sqrt{3}$

Taking greater slope of CE as

$2+\sqrt{3}$

$y-1=\left(2+\sqrt{3}\right)(x-1)$

Put y = 0

$-1=\left(2+\sqrt{3}\right)(x-1)$

$\frac{-1}{2+\sqrt{3}}\times\left(\frac{2-\sqrt{3}}{2-\sqrt{3}}\right)=x-1$

$x-1=\sqrt{3}-1$

EB = $1-x=1-\left(\sqrt{3}-1\right)$

$EB=2-\sqrt{3}$

4. If $\lim_{x\to 0}\frac{ae^x-b\cos x+ce^{-x}}{x\sin x}=2,$ then a + b + c is equal to ________ ,

Ans. 4

Solution:

$$\lim_{x\to 0}\frac{ae^{x}-b\cos x+ce^{-x}}{x\sin x}=2$$

$$\Rightarrow \lim_{x\to 0}\frac{a\left(1+x+\frac{x^2}{2!}....\right)-b\left(1-\frac{x^2}{2!}+....\right)+c\left(1-x+\frac{x^2}{2!}\right)}{\left(\frac{x\sin x}{x}\right)x}=2$$

$a - b + c = 0$ ...(1)

$a - c = 0$ ....(2)

& $\frac{a+b+c}{2}=2$

$\Rightarrow \boxed{a+b+c=4}$

5. The total number of 3×3 matrices A having entries from the set {0, 1, 2, 3} such that the sum of all diagonal entries of AA^T is 9, is equal to ________ .

Ans. 766

Solution:

Let A = $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

diagonal elements of

AAT, $a^2 + b^2 + c^2$, $d^2 + e^2 + f^2$, $g^2 + b^2 + c^2$

Sum = $a^2 + b^2 + c^2 + d^2 + e^2 + f^2 + g^2 + h^2 + i^2 = 9$

a, b, c, d, e, f, g, h, i $\in$ {0, 1, 2, 3}

	Case	No. of Matrices
(1)	All – 1s	$\frac{9!}{9!}=1$
(2)	One $\to$3 remaining-0	$\frac{9!}{1!\times 8!}=9$
(3)	One-five-1s three-0s	$\frac{9!}{1!\times 5!\times 3!}=8\times 63$
(4)	two-2's one-1 six-0's	$\frac{9!}{2!\times 6!}=63\times 4$

Total no. of ways = $1 + 9 + 8\times 63 + 63\times 4 = \boxed{766}$

6. Let $P=\begin{bmatrix} -30 & 20 & 56 \\ 90 & 140 & 112 \\ 120 & 64 & 14 \end{bmatrix}$ and $A=\begin{bmatrix} 2 & 7 & \omega^2 \\ -1 & -\omega & 1 \\ 0 & -\omega & -\omega+1 \end{bmatrix}$ where $\omega=\frac{-1+i\sqrt{3}}{2}$, and I_3 be the identity matrix of order 3. If the determinant of the matrix $(P^{-1}AP - I_3)^2$ is $\alpha\omega^2$, then the value of α is equal to ____ .

Ans. 36

Solution:

Let $M = (P^{-1}AP - I)^2$

$= (P^{-1}AP)^2 - 2P^{-1}AP + I$

$= P^{-1}A^2P - 2P^{-1}AP + I$

$PM = A^2P - 2AP + P$

$= (A^2 - 2A.I + I2)P$

$\Rightarrow Det(PM) = Det((A - I)^2 \times P)$

$\Rightarrow$ Det P. Det M = $Det(A - I)^2 \times Det(P)$

$\Rightarrow$ Det M = $(Det\,(A - I))^2$

Now $A-I=\begin{bmatrix} 1 & 7 & w^2 \\ -1 & -w-1 & 1 \\ 0 & -w & -w \end{bmatrix}$

$Det\,(A - I) = (w^2 + w + w) + 7(-w) + w^3 = -6w$

$Det\,((A - I))^2 = 36w^2$

$\Rightarrow \alpha = 36$

7. If the normal to the curve $y(x)=\int_0^x (2t^2-15t+10t)\,dt$ at a point (a, b) is parallel to the line $x + 3y = -5$, $a > 1$, then the value of $|a+6b|$ is equal to _________ .

Ans. 406

Solution:

$y(x)=\int_0^x (2t^2-15t+10)dt$

$y'(x)\,]_{x=a}=\left[2x^2-15x+10\right]_a = 2a^2-15a+10$

Slope of normal = $-\frac{1}{3}$

$\Rightarrow 2a^2-15a+10=3 \Rightarrow a=7$

& $a = \frac{1}{2}$ (rejected)

$b = y(7) = \int_0^7 (2t^2-15t+10)dt$

$= \left[\frac{2t^3}{3}-\frac{15t^2}{2}+10t\right]_0^7$

$\Rightarrow 6b = 4\times 7^3 - 45\times 49 + 60\times 7$

$|a+6b| = 406$

8. Let the curve y = y(x) be the solution of the differential equation, $\frac{dy}{dx} = 2(x+1)$. If the numerical value of area bounded by the curve y = y(x) and x-axis is $\frac{4\sqrt{8}}{3}$, then the value of y(1) is equal to_____ .

Ans. 2

Solution:

$\frac{dy}{dx} = 2(x+1)$

$\Rightarrow \int dy = \int 2(x+1)dx$

$\Rightarrow y(x) = x^2 + 2x + C$

Area = $\frac{4\sqrt{8}}{3}$

$-1+\sqrt{1-C}$

$\Rightarrow 2\int_{-1}^{-1+\sqrt{1-C}} \left(-(x+1)^2 - C + 1\right)dx = \frac{4\sqrt{8}}{3}$

$\Rightarrow 2\left[-\frac{(x+1)^3}{3} - Cx + x\right]_{-1}^{-1+\sqrt{1-C}} = \frac{4\sqrt{8}}{3}$

$\Rightarrow -\left(\sqrt{1-C}\right)^3 + 3c - 3C\sqrt{1-C}$

$-3+3\sqrt{1-C} - 3C + 3 = 2\sqrt{8}$

$\Rightarrow$ C = –1

$\Rightarrow f(x) = x^2 + 2x - 1,\ f(1) = 2$

9. Let f: $R \to R$ be a continuous function such that f(x) + f(x + 1) = 2, for all $x \in R$.

If $I_1 = \int_0^8 f(x)dx$ and $I_2 = \int_{-1}^3 f(x)dx$, then the value of $I_1 + 2I_2$ is equal to __________ .

Ans. 16

Solution:

f(x) + f(x + 1) = 2

$\Rightarrow$ f(x) is periodic with period = 2

$I_1 = \int_0^8 f(x)dx = 4\int_0^2 f(x)dx$

$= 4\int_0^1 \left(f(x) + f(1+x)\right)dx = 8$

Similarly $I_2 = 2\times 2 = 4$

$I_1 + 2I_2 = 16$

10. Let z and w be two complex numbers such that $w = z\bar{z} - 2z + 2, \left|\frac{z+i}{z-3i}\right| = 1$ and Re(w) has minimum value. Then, the minimum value of $n \in N$ for which w^n is real, is equal to ________ .

Ans. 4

Solution:

Diagram pg 10 Que 10

$\omega = z\bar{z} - 2z + 2$

$\left|\frac{z+i}{z-3i}\right| = 1$

$\Rightarrow |z+i| = |z-3i|$

$\Rightarrow z = x + i, x \in R$

$\omega = (x+i)(x-i) - 2(x+i) + 2$

$= x^2 + 1 - 2x - 2i + 2$

$\text{Re}(\omega) = x^2 - 2x + 3$

For min $(\text{Re}(\omega))$, $x = 1$

$\Rightarrow \omega = 2 - 2i = 2(1-i) = 2\sqrt{2}\, e^{-i\frac{\pi}{4}}$

$\omega^n = (2\sqrt{2})^n\, e^{-i\frac{n\pi}{4}}$

For real & minimum value of n, n = 4