Regional Mathematical Olympiad- 2017 Solutions

Time: 3 hours **October 08, 2017**

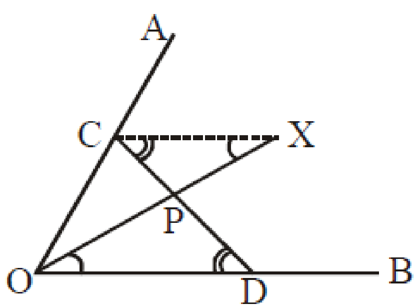
Instructions:

- Calculators (in any form) and protractors are not allowed.
- Rulers and compasses are allowed.
- Answer all the questions.
- All questions carry equal marks. Maximum marks: 102.

Answer to each question should start on a new page. Clearly indicate the question number.

1. Let AOB be a given angle less than 180^0 and let P be an interior point of the angular region determined by $\angle AOB$. Show with proof, how to construct, using only ruler and compasses, a line segment CD passing through P such that C lies on the ray OA and D lies on the ray OB, and CP : PD = 1 : 2.

Solution:



Just extend OP to X, such that OP : PX = 2:1. Draw a line through X parallel to OB which meets OA at C. Extend CP to meet OB at D. CD is the required line.

$\because \ \Delta CPX \sim \Delta DPO$

$$\frac{CP}{DP} = \frac{PX}{PO} = \frac{1}{2}$$

$\therefore$ PC : PD = 1 : 2

2. Show that the equation

$a^3 + (a+1)^3 + (a+2)^3 + (a+3)^3 + (a+4)^3 + (a+5)^3 + (a+6)^3 = b^4 + (b+a)^4$ has no solutions in integers a, b.

Solution:

Since 7 consecutive numbers appear on left side, it's a good idea to try modulo 7.

$1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 \equiv 1 + 1 + (-1) + 1 + (-1) + (-1) \equiv 0 \pmod 7$

So LHS is always divisible by 7.

Or, $a^3 + (a+1)^3 + ...(a+6)^3 \equiv \sum_{r=1}^{7} r^3$ (mod 7) = $\left[\frac{7(7+1)}{2}\right]^2$ (mod 7) = $(28)^2$ (mod 7) = 0 (mod 7)

Now RHS, $b^4 + (b+1)^4$

for any integral b,

$b \equiv r$ (mod 7), where r is 0, 1, 2, 3, 4, 5, 6

$b^4 + (b+1)^4 \equiv r^4 + (r+1)^4 \not\equiv 0$ (mod 7)

for any r = 0, 1, 2, 3, 4, 5, 6

Hence, no solution.

3. Let $P(x) = x^2 + \frac{1}{2}x + b$ and $Q(x) = x^2 + cx + d$ be two polynomials with real coefficients such that $P(x)Q(x) = Q(P(x))$ for all real x. Find all the real roots of $P(Q(x)) = 0$

Solution:

$P(x) = x^2 + \frac{x}{2} + b$

$Q(x) = x^2 + cx + d$ ...(i)

$P(x).\ Q(x) = Q(P(x))$

let us consider $P(x) = 0$

$0 = Q(0) = d$ (from (i))

$\Rightarrow d = 0$

Now $P(x).\ Q(x) = [P(x)]^2 + cP(x)$

$\Rightarrow Q(x) = [P(x)] + c$

$\Rightarrow x^2 + cx = x^2 + \frac{x}{2} + b + c$

$\Rightarrow c = \frac{1}{2}$ and $b + c = 0$

$\Rightarrow b = -\frac{1}{2}$

$\Rightarrow P(x) = x^2 + \frac{x}{2} - \frac{1}{2}$ and $Q(x) = x^2 + \frac{1}{2}x$

Now since $P(Q(x)) = 0$, $Q(x)$ is a root of $P(x) = 0$

i.e. $x^2 + \frac{x}{2} - \frac{1}{2} = 0 \quad \Rightarrow (x+1)\left(x - \frac{1}{2}\right) = 0 \quad \Rightarrow x = -1, \frac{1}{2}$

$\therefore$ Q(x) has to be either -1 or $\frac{1}{2}$

Case 1:

$Q(x) = -1$

$$\Rightarrow x^2 + \frac{1}{2}x + 1 = 0$$

$$\Rightarrow x = \frac{-1 \pm \sqrt{15}\,i}{4}$$

Case 2:

$$Q(x) = \frac{1}{2} \Rightarrow x^2 + \frac{1}{2}x - \frac{1}{2} = 0$$

$$\Rightarrow x = \frac{1}{2}, -1$$

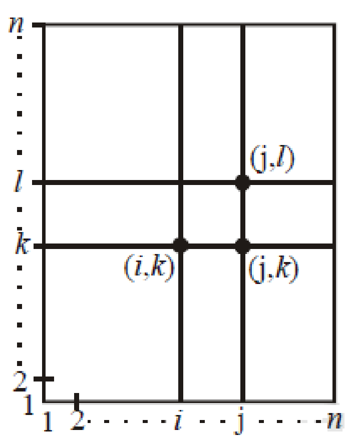
So total 4 roots, 2 real and 2 imaginary.

Real roots are $\frac{1}{2}$ & -1

4. Consider n^2 unit squares in the xy-plane centred at point (i, j) with integer coordinates, $1 \le i \le n, 1 \le j \le n$. It is required to colour each unit square in such a way that whenever $1 \le i < j \le n$ and $1 \le k < \ell\ n$, the three squares with centres at (i, k), (j, k), (j, l) have distinct colours. What is the least possible number of colours needed?

Solution:

Here $1 \le i < i \le n$; $1 \le k < 1 \le n$



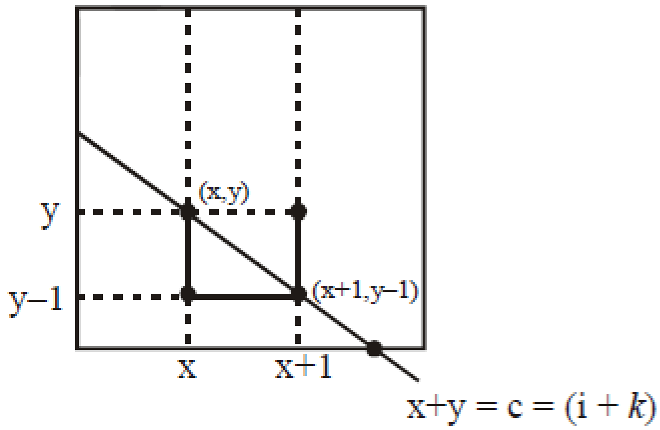
as per given condition, (i, k), (j, k), (j, l) are of distinct colours.

$\Rightarrow$ No two square of k^{th} row will be of same color as (i, k) and (j, k) are of distinct colors and no to squares of j^{th} column of same color as (j, l) and (j, k) are of distinct color

$\Rightarrow$ all square of type (x, 1) for x = 1, 2.............n, are of distinct color $\Rightarrow$ total n distinct colours

Similarly all square of type (n, y) for y = 1, 2.......n, are of distinct colors also no square of (x, 1) and (n, y) with same color otherwise take square (n, n) with (x, 1) and (n, y) we will get contradiction to given condition.

There will be at least 2n – 1 colors. Now let us prove 2n – 1 colours are sufficient.



We can see that (x, y) and(x + 1, y – 1) can have same colours. So paint all squares with same colour for which x + y is same.

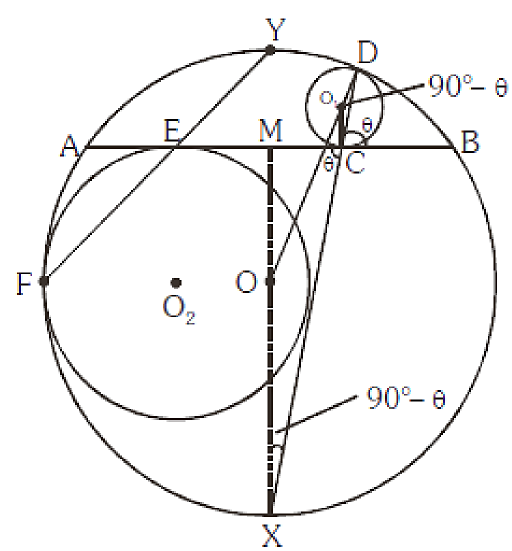
Here minimum x + y will be 2 (for x = 1, y = 1) and maximum x + y = 2n

As from 2 to 2n there 2n – 1 values, so 2n – 1 colours will be sufficient.

5. Let Ω be a circle with a chord AB which is not a diameter. Let T_1 be a circle on one side of AB such that it is tangent to AB at C internally tangent to Ω at D. Likewise, let T_2 be a circle on the other side of AB such that it is tangent to AB at E and internally tangent to Ω at F. Suppose the line DC intersects Ω at $X \neq D$ and the line FE intersects Ω at $Y \neq F$. Prove that XY is a diameter of Ω.

Solution:

Let O_1 be centre of Γ_1, O of Ω and O_2 of Γ_2. Join OX and extend it to meet AB at M. Join OO_1 it will pass through D (as both circles are tangent)



Now let $\angle DCB = \theta$

$\Rightarrow \quad \angle O_1CD = 90^0 - \theta$

$\Rightarrow \quad \angle O_1DC = 90^0 - \theta$

$\Rightarrow \quad \angle ODX = 90^0 - \theta$

$\Rightarrow \quad \angle OXD = \angle ODX = 90^0 - \theta$

or $\quad \angle MXC = 90^0 - \theta$

Also $\quad \angle MCX = \angle DCB = \theta$ (VOA)

In ΔXCM

$\angle CMX = 180^0 - \angle MXC \;\; \angle MCX$

$= 180^0 - (90^0 - \theta) - \theta = 90^0$

$\Rightarrow$ X is mid-point of arc AB (not containing D) similarly Y is mid-point of arc AB (containing D)

$\Rightarrow$ XY is a diameter of Ω.

6. Let x, y, z be real numbers, each greater than 1. Prove that

$$\frac{x+1}{y+1}+\frac{y+1}{z+1}+\frac{z+1}{x+1}\le\frac{x-1}{y-1}+\frac{y-1}{z-1}+\frac{z-1}{x-1}$$

Solution:

$$\frac{x-1}{y-1}+\frac{y-1}{z-1}+\frac{z-1}{x-1}\ge\frac{x+1}{y+1}+\frac{y+1}{z+1}+\frac{z+1}{x+1}$$

$$\frac{x-1}{y-1}+\frac{x+1}{y+1}+\frac{y-1}{z-1}-\frac{y+1}{z+1}+\frac{z-1}{x-1}-\frac{z+1}{x+1}\ge 0$$

$$\frac{2(x-y)}{y^2-1}+\frac{2(y-z)}{z^2-1}+\frac{2(z-x)}{x^2-1}\ge 0$$

$$\frac{x-y}{y^2-1}+\frac{y-z}{z^2-1}+\frac{z-x}{x^2-1}\ge 0$$

i.e., $\frac{x}{y^2-1}-\frac{y}{y^2-1}+\frac{y}{z^2-1}+\frac{z}{z^2-1}+\frac{z}{x^2-1}-\frac{x}{x^2-1}\ge 0$

$$\frac{x}{y^2-1}+\frac{y}{z^2-1}+\frac{z}{x^2-1}\ge\frac{y}{y^2-1}+\frac{z}{z^2-1}+\frac{x}{x^2-1}$$

Now, without loss of generality, let

$$z\ge y\ge x>1$$

$\Rightarrow$ $z^2-1\ge y^2-1\ge x^2-1>0$

$\Rightarrow$ $\frac{1}{x^2-1}\ge\frac{1}{y^2-1}\ge\frac{1}{z^2-1}>0$

Now, applying rearrangement inequality,

$$\begin{bmatrix} x & y & z \\ \frac{1}{x^2-1} & \frac{1}{y^2-1} & \frac{1}{z^2-1}\end{bmatrix}\le\begin{bmatrix} x & y & z \\ \frac{1}{y^2-1} & \frac{1}{z^2-1} & \frac{1}{x^2-1}\end{bmatrix}$$

$\Rightarrow$ $\frac{x}{x^2-1}+\frac{y}{y^2-1}+\frac{z}{z^2-1}\le\frac{x}{y^2-1}+\frac{y}{z^2-1}+\frac{z}{x^2-1}$

Alternate 1:

$$\frac{x-y}{y^2-1}+\frac{y-z}{z^2-1}+\frac{z-x}{x^2-1}\geq 0$$

let without loss of generality,

$$x \geq y \geq z$$

$$x^2 \geq y^2 \geq z^2$$

$$x^2-1 \geq y^2-1 \geq z^2-1$$

$$\frac{1}{z^2-1} \geq \frac{1}{y^2-1} \geq \frac{1}{x^2-1}$$

let $\frac{1}{z^2-1}=a, \frac{1}{y^2-1}=b, \frac{1}{x^2-1}=c$

$\therefore$ $a \geq b \geq c$

It is suffices to show that

$$\frac{x-y}{y^2-1}+\frac{y-z}{z^2-1}\geq\frac{x-z}{x^2-1}$$

i.e. $b(x-y)+a(y-z) \geq c(x-z)$

which is true by adding the following

$b(x-y) \geq c(x-y)$

$a(y-z) \geq c(y-z)$