SET II
SECTION -A

1. If x, y, z be positive numbers, show that

$(x + y + z)^3 \geq 27\ xyz.$ [4 marks]

Sol : since A.M. (arithmetic mean)≥ G.M. (geometric mean), therefore

$$\frac{x+y+z}{3} \geq (xyz)^{1/3.}$$

Cubing both sides and multiplying throught by 27, we have

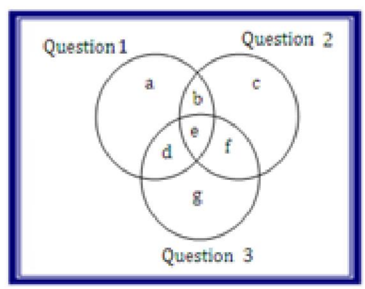
$(x+y+z)^3 \geq 27\ xyz.$

2. In an examination, question 1 was attempted by 87 candidates, question 2 by 66 candidates and question 3 by 60 candidates. 40 candidates attempted both questions 1 and 2, 17 attempted both questions 2 and 3, 37 attempted both questions 1 and 3, and 5 attempted all the three questions :

(i) How many attempted question 1 but not 2 and 3?

(ii) How many attempted question 2 but not 1 and 3 ? [4 marks]

Sol:



a+b+d+e =87....(1)

b+c+e+f=66....(2)

d+e+f+g=60....(3)

b+e=40..... (4)

e+f=17.....(5)

d+e= 37.....(6)

e=5...(7)

(i) a=?

from 4 & 7

b=35.........(8)

from 5 & 7

f=12....(9)

from 6&7

d=32....(10)

from 1,7,8,10, we get

a+35+32+5=87

a=87-72=15

a=15

(ii) c=?

from 2,7,8 &9 we get

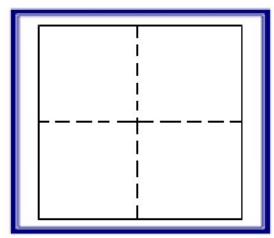
35+c+5+12=66

c=66-52=14

c=14

3. Five points lie within a square, each of whose sides has length 2 units. Show that there exist two of them the distance between which is at the most $\sqrt{2}$.

Sol :



Divide the given square into four equal squares by means of dotted lines as in Figure.

By pigeon hole principle, at least two of then (say, x and y) must lie in one of the four small squares. The distance between x and y cannot exceed the length of the diagonal of the square, i.e., $\sqrt{2}$.

[4 marks]

4. 29th February of the year 2000 will fall on a Tuesday, for your information. Show that, after this date, 29th February will fall on Tuesday only thrice in the whole next century. What are the three years when this will happen ?

Sol :

Since 365 $\equiv 1$ (mod. 7), therefore 28th February (the last day of February) of 2001 will be a Wednesday (the day next to Tuesday). Let us agree to express this by saying that there is an excess of one day any ordinary year. With this terminology, there will be an excess of two days in a leap year.

The next leap year after the year 2000 will be the year 2004. There will be 1 + 1 + 1 + 2, i.e., 5 excess days upto 29th February , 2004. The day of the week of 29th February will be Tuesday when the number of excess day is an multiple of 7. Therefore our problem is to find those positive integers k for which the number of excess days in 4k years after the year 2000 is an exact multiple of 7 and for which 4k < 100. The number of excess days in 4k years is 5k. This is a multiple of 7 when k = 7, 14, 21, ….., i.e., in 4 × 7, l4 × 14 × 21, … years after the year 2000. Since only three of these numbers are less than 100, therefore in the 21st century there are only three years, namely 2028, 2056 and 2084 in which 29th February is a Tuesday

[4 marks]

5. In a certain town, there are ten restaurants and n theatres. A group of tourists spent a few days in the town and visited the theatres and the restaurants during their stay. At the end of their stay it turned out that in all, each restaurant was visited exactly by 4 tourists and that each theatre was visited by 6 tourists. Given that each tourist visited exactly 5 restaurants and 3 theatres, find n, the number of theatres.

Sol :

Let there be m tourists in all. Counting each visit of a tourist separately, we find that the total number of visits by all the tourists to all the restaurants = 10 × 4 (number of restaurants × number of tourists visiting each restaurant). This number is the same as m × 5 (number of tourists × number of restaurants visited by each tourists) = 5m.

Since 5m = 40, therefore m = 8.

Proceeding as above we find that

6 × n = 3 × 8,

i.e., n = 4.

Therefore there are 4 theatres in the town

[4 marks]

6. The sum of two positive integers is 52 and their LCM is 168. Find the numbers.

[4 marks]

Sol :

Let the GCD of the numbers be d. Then the numbers must be of the form dp, dq, where p, q are prime to each other.

We are given that the sum of the numbers is 52. This gives

dp + dq = 52,

i.e., d (p + q) = 52. ...(i)

Also, since the LCM of the numbers is 168, therefore

dpq = 168 ...(ii)

from (i) and (ii), we find that d is a common divisor of 52 cards 168. Since GCD of 52 and 168. Since GCD of 52 and 168 is 4, therefore d must have one of the values 1, 2 or 4.

If d = 1, p + q = 52, pq = 168. Since p, q are relatively prime, therefore the only possible values for the pair (p, q) are (1, 168), (8, 21), (24, 7), (56, 3), none of which has the sum 52. Therefore we conclude that d ≠ 1. If d = 2, the possible values for the pair (p, q) are (1, 84), (3, 28), (4, 21), (7, 12), and since none of these satisfies the condition d (p + q) = 52, therefore it follows that d ≠ 2.

If d = 4, then p + q = 13, pq = 42. The possible values for the pair (p, q) satisfying the condition pq = 42 are (1, 42), (2, 21), (3, 14), (6, 7).

The only one pair satisfying the condition p + q = 13 is (6, 7). Therefore the numbers are 6 × 4 and 7 × 4, i.e., 24 and 28.

Aliter:

From (i) and (ii), we have

$$\frac{p+q}{pq}=\frac{13}{42}.$$

We have to find integer solutions of the equation

$$\frac{1}{p}+\frac{1}{q}=\frac{13}{42}.$$

Without loss of generality let us assume that p > q.

Then, $\frac{1}{p}<\frac{1}{p}+\frac{1}{q}=\frac{13}{42},$

so that $p\geq\frac{42}{13},$

i.e., $p\geq 4.$

Also $\frac{1}{p}+\frac{1}{q}<\frac{2}{p},$

so that $\frac{13}{42}<\frac{2}{p},$

which gives $p<\frac{84}{13},$

so that $p\leq 6.$

Thus $4\leq p\leq 6.$

For p = 4, 5 we do not get any integral value of q. For p = 6, we get q = 7. Therefore the desired are 6 × 4, 7 × 4, i.e., 24, 28 as before.

Aliter. From (i) and (ii) above, we have

42(p +q) = 13 pq.

Since p and q are prime to each other, therefore p and q are both prime to p + q, i.e., pq is prime to p + q. Therefore pq divides 42.

Also 42 being prime to 13, it follows that | pq.

Since the positive integers pq and 42 divide each other, therefore pq = 42, and consequently p + q = 13.

Therefore p, q are the roots of the equation $x^2 - 13x + 42 = 0$, so that p, q = 6, 7. This gives $d = \frac{168}{pq} = 4$, so that the numbers are 24, 28 as before.

7. ABCD is a parallelogram, P, Q, R and S are points on sides AB, BC, CD and DA respectively such that AP = DR. If the area of parallelogram ABCD is 16 cm^2, find the area of the quadrilateral PQRS.

Sol :

Since AP = DR, and AP || DR, therefore APRD is a $||^{gm}$, and consequently PR || AD.

Since ΔPRS and $||^{gm}$ PRDA have the same altitude, therefore area of $\Delta PRS = \frac{1}{2} \times$ area of $||^{gm}$ PRDA.

Again, since ΔPRQ and $||^{gm}$ PRCB have the same base PR, and have the same altitude, therefore $\Delta PQR = \frac{1}{2} \times$ area of $||^{gm}$ PBCR.

Area of quad. PSRQ

= Area of ΔPSR + area of ΔPQR

$= \frac{1}{2} \times \text{area of } ||^{gm} \text{ PRDA} + \frac{1}{2} \times \text{area of } ||^{gm} \text{ PRCB},$

$= \frac{1}{2} \times \text{area of } ||^{gm} \text{ ABCD},$

$= \frac{1}{2} \times 16 \text{ cm}^2 = 8 \text{ cm}^2.$

[4 marks]

8. Each of the three girls Sheela, Leela and Sakeena, has in her purse exactly one of the following objects, a pencil, a ball-pen and an eraser. Out of the following statements one is true and two are false:

1. Sheela has the pencil;

2. Leela does not have the pencil;

3. Sakeena does not have the eraser.

Determine who has which object.

Sol :

Suppose statement 1 is true. Then Sheela has the pencil, and therefore Leela does not have the pencil, which means that statement 2 is also true. However, both statements 1 and 2 cannot be true. Therefore statement 1 is false

Suppose statement 2 is true. Then Leela does not have the pencil. Also Sheela does not have the pencil. Consequently Sakeena has the pencil, and therefore statement 3 is true. Since both statements 2 and 3 cannot be true, therefore statement 2 is false.

Since statements 1 and 2 are both false, therefore statement 3 must be true. This means that Sakeena does not have either the eraser or the pencil, which implies that Sakeena has the ball-pen.

Since Leela has the pencil and Sakeena has the ball-pen, therefore Sheela has the erase,

Thus Sheela has the eraser, Leela has the pencil, and Sakeena has the ball-pen

[4 marks]

9. Find the number of divisors of 7056.

Sol :

$7056 = 2^4 . 3^2 . 7^2$.

Number of divisors = (4 + 1)(2 + 1)(2 + 1) = 45

[4 marks]

10. Factorize $x^6 + 5x^3 + 8$.

Sol :

The given expression can be written as

$(x^2)^3 + (-x)^3 + (2)^3 - 3. x^2 . (- x) . 2$,

which is of the form $a^3 + b^3 + c^3 - 3abc$, with $a = x^2$, $b = - x$, $c = 2$.

Therefore the desired factorization is

$(a + b + c) (a^2 + b^2 + c^2 - bc - ca - ab)$,

$= (x^2 - x + 2) (x^4 + x^2 + 4 + 2x - 2x^2 + x^3)$,

$= (x^2 - x + 2) (x^4 + x^3 - x^2 + 2x + 4)$.

[4 marks]

Section – B

11. If $(x + 1)^2$ is greater than $5x - 1$ and less than $7x - 3$, find the integral values of x.

Sol :

We are given that

$(x + 1)^2 > 5x - 1$ and $(x + 1)^2 < 7x - 3$,

i.e., $x^2 - 3x + 2 > 0$ and $x^2 - 5x + 4 < 0$,

i.e., $(x - 1)(x - 2) > 0$ and $(x - 1)(x - 4) < 0$.

i.e., (either $x < 1$ or $x > 2$) and ($x > 1$ and $x < 4$).

Two different cases arise :

Case I. $x < 1$ and ($x > 1$ and $x < 4$).

Obviously there is not real number x which satisfies $x < 1$ as well as $x > 1$,

Therefore this possibility is not true.

Case II. $x > 2$ and ($x > 1$ and $x < 4$). Clearly we must have $x > 2$ and $x < 4$. The integral value of x for which $x > 2$ and $x < 4$ holds is $x = 3$, which is the desired integral value

[6 marks]

12. Show that if $a + b + c = 0$, then

$$\left(\frac{b-c}{a}+\frac{c-a}{b}+\frac{a-b}{c}\right)\left(\frac{a}{b-c}+\frac{b}{c-a}+\frac{c}{a-b}\right)=9.$$

Sol :

$$\left(\frac{b-c}{a}+\frac{c-a}{b}+\frac{a-b}{c}\right)=\frac{bc(b-c)+ca(c-a)+ab(a-b)}{abc}$$

$$=-\frac{(b-c)(c-a)(a-b)}{abc}. \qquad \text{.........(1)}$$

Also, $\frac{a}{b-c}+\frac{b}{c-a}+\frac{c}{a-b}$

$$= \frac{a(c-a)(a-b)+b(b-c)(a-b)+c(b-c)(c-a)}{(b-c)(c-a)(a-b)},$$

$$= \frac{\sum\{-a^3+a^2(b+c)-abc\}}{\{(b-c)(c-a)(a-b)\}},$$

$$= \frac{\{-2(a^3+b^3+c^3)-3abc\}}{\{(b-c)(c-a)(a-b)\}},$$

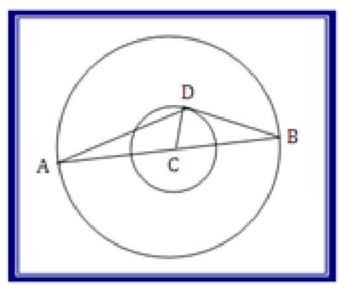
since a + b + c = 0

$$\frac{\{-2(3abc)-3abc\}}{\{(b-c)(c-a)(a-b)\}}$$

$$= -\frac{9\,abc}{(b-c)(c-a)(a-b)}. \qquad ....(2)$$

Multiplying the corresponding sides of (1) and (2), we immediately get the desired result [6 marks]

13. The radii of two circles in a plane are 13 and 18. Let AB be a diameter of the larger circle and BC a chord of the larger circle tangent to the smaller circle at D. Calculate the length of AD.



Sol :

Since CB = 13 cm, CD = 8 cm, therefore

BD = $\sqrt{(13^2-8^2)} = \sqrt{105}$ cm.

In Δ ADB, C is the mid-point of AB. By Apollonius theorem, $AD^2 + BD^2 = 2\,(AC^2 + CD^2)$, so that $AD^2 = 2\,(13^2 + 8^2) - 105, = 361$, so that AD = 19 cm [6 marks]

14. Five men A, B, C, D, E are wearing caps of black or white colour without each knowing the colour of his cap. It is known that a man wearing a black cap always speaks the truth while the ones wearing white tell lies. If they make the following statements, find the worn by each of them :

A: I see three black caps and one white.

B: I see four white caps.

C: I see one black cap and three white.

D: I see four black caps

Sol :

Suppose A is speaking the truth, Then he must he wearing a black cap, so that four persons are wearing black caps and one person is wearing a white cap. The statements made by B and C are both in contradiction with this statement. Therefore B and C must be wearing white caps. This implies that A' s statement is false. The contradiction show that A is telling a lie, and therefore A is wearing a white cap. Suppose now that B is speaking the truth. Then B must be wearing a black cap, and the caps of C, D, E must be all white. Since B's cap is black and caps of A, D, E are white, therefore C's statement is true and consequently his cap must be black. We have a contradiction, which shows that B is telling a lie, and therefore B is wearing a white cap.

Since A and B are both wearing white caps, therefore D is telling a lie. Therefore we find that D is wearing a white cap.

If C is speaking the truth, then E's cap must be black and C's own cap must also be black . If on the other hand, C is telling a lie, then E's cap must be white (because that is the only way in which his statement can be false) and C's cap must also be white. This implies all the five caps are white. But this means that B is speaking the truth and so his cap must be black. The contradiction shows that C's statement cannot be false. Thus C is speaking the truth, and the caps of C and E must be both black.

Hence A, B, D are wearing white caps, and C, E are wearing black caps.

Aliter:

Since E has not made any statement, we shall start with the colour of E's cap. Two different cases arise : E's cap is either white or black.

Let us first suppose that E is a wearing a white cap.

Since D cannot see four black caps (at least E's cap is white), therefore D must be telling a lie and colour of his cap must be white.

Since D and E have white caps, A's statement is false and his cap is also white.

Suppose now that B's cap is black. Then he is speaking the truth and C's cap is white. But the caps of A, D, E being white and that of B being black (he is speaking the truth), C's statement is true and his cap must be black. This is a contradiction and consequently B's cap must be white.

If B's cap is white , he is telling a lie and so C's cap must be black (since caps of A, D, E are already white). Since C's cap is black, B's cap must be black because C sees one black cap. We again have a contradiction and consequently B's cap cannot be white. From the a contradiction and consequently B's cap cannot be white. From the above we conclude that E's cap cannot be white.

Now E's cap being black, B cannot see four white caps, and consequently his own cap must be white.

Since B's cap is white, D cannot see four black caps and therefore his own cap must be white.

Since caps of B and D are white, A cannot see three black caps. Consequently his own cap must be white.

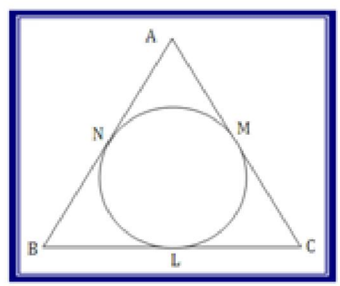
Now caps of A, B, D are white and that of E is black, therefore C's statement is true and his own cap must be black.

Thus A, B and D are wearing white caps, and C and E are wearing black caps

[6 marks]

15. The radius of the incircle of a triangle is 24 cm, and the segments into which one side is divided by the point of contact are 36 cm and 48 cm. Find the lengths of the other two sides of the triangle.

Sol:



Let the sides BC, CA and AB touch the incircle at the points L, M, N respectively.

Suppose BC is divided by L into segments BL, LC measuring 36 cm and 48 cm respectively.

Since, BL, BN are tangents to the incircle from B touching it at L and N respectively, therefore BL = BN = 36 cm.

Similarly CL = CM = 48 cm.

Also AM = AN =x cm (say).

With the usual notation,

a = (36 + 48) cm = 84 cm, b = (48 + x) cm,

$c = (x + 36)$ cm, $s = \frac{1}{2}(a + b + c) = (x + 84)$ cm,

$s - a = x$ cm, $s - b = 36$ cm, $s - c = 48$ cm,

$\Delta = \sqrt{\{s(s-a)(s-b)(s-c)\}} = \sqrt{\{(x+84).x.36.48\}}\ \text{cm}^2.$

Now $\quad r = \frac{\Delta}{s} = \frac{\sqrt{\{(x+84).x.36.48\}}}{x+84} = 24\left\{\frac{3x}{x+84}\right\}$

Since r = 24, therefore, we get

$$\frac{24}{\left\{\frac{3x}{x+84}\right\}^{1/2}} = 24,$$

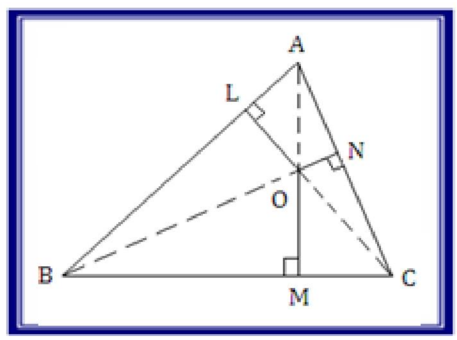
so that x = 42. This gives b = 90 cm, c = 78 cm

[6 marks]

16. Let O be any point inside a triangle ABC. Let L, M and N be points on AB, BC and CA respectively where the perpendiculars from O meet these lines. Show that

$AL^2 + BM^2 + CN^2 = AN^2 + CM^2 + BL^2$

Sol :



Join OA, OB and OC.

In rt. $\angle$d triangles OAL, OBM, OCN

$AL^2 = OA^2 - OL^2$

$BM^2 = OB^2 - OM^2,$

$CN^2 = OC^2 - ON^2,$

so that, $\quad AL^2 + BM^2 + CN^2 = (OA^2 + OB^2 + OC^2) - (OL^2 + OM^2 + ON^2)$ ……(1)

Again, in rt., $\angle$d Δs OAN, OBL, OCM,

$AN^2 = OA^2 - ON^2,$

$BL^2 = OB^2 - OL^2$

$CM^2 = OC^2 - OM^2,$

so that, $AN^2 + BL^2 + CM^2 = (OA^2 + OB^2 + OC^2) - (OL^2 + OM^2 + ON^2)$ ……..(2)

From (1) and (2) we find that

$AL^2 + BM^2 + CN^2 = AN^2 + CM^2 + BL^2$

[6 marks]

17. A man went out between 2 and 3 O'clock and came back between 4 and 5 O'clock. He found that the hands had exactly changed their places. At what time did he go out?

Sol :

When the man went out, when the hour and was between 2 and 3, and the minute hand was between 4 and 5. When he came back the hour hand was between 4 and 5, and the minute hand was between 2 and . Since the hands have exactly changed places, it follows that the two hands together must have made 2 complete revolutions i.e., they must have moved through 120 minute-divisions.

The minute hand moves 12 times as fast as the tour hand.

$\therefore$ During this time the hour hand must have moved through $\frac{120 \times 1}{13}$ minute-divisions, i.e., $9\frac{3}{13}$ minute-divisions.

$\therefore$ When the man went out, the minute hand must have been $9\frac{3}{13}$ minute-divisions ahead of the hour hand.

At 2 O'clock the minute hand was 10 minute-divisions behind the hour hand.

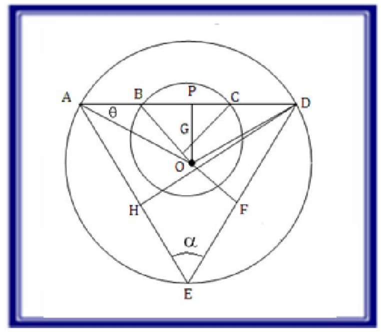
$\therefore$ Time required by the minute hand to gain $\left(10+9\frac{3}{13}\right)$ minute-divisions over the hour hand = $\frac{250}{13} \times \frac{12}{11}$ minutes = $20\frac{140}{143}$ minutes.

$\therefore$ The man must have gone out at $20\frac{140}{143}$ minutes past 2

[6 marks]

18. A straight line meets two concentric circles in points A, B, C and D in that order. AE and BF are parallel chords, one in each circle. If CG ⊥ BF and DH ⊥ AE, prove that GF = HE.

Sol :



Let P be the foot of the perpendicular from O on AD, and p the length of the perpendicular OP. Also, let $\angle PAE = \theta$.

$\angle AED = \frac{1}{2}\angle AOD = \alpha.$

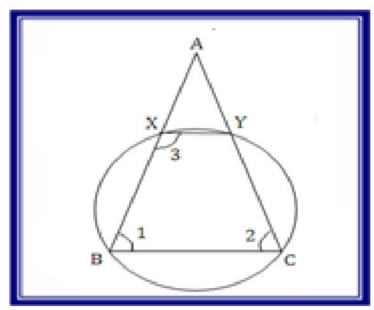
Also, in ΔADE, $\frac{DE}{\sin\theta} = \frac{AD}{\sin\alpha}$.

Also, AD = 2R sin α, where R is the radius of the bigger circle. Thus DE = 2R sin θ. In rt. ∠d ΔDHE, HE = DE cos∝ = 2R sin θ cos α = 2p sin θ. Hence GF = HE

[6 marks]

19. Δ ABC is an isosceles triangle, and XY is drawn parallel to the base cutting the sides in X and Y. Show that the four points B, C, X, Y lie on a circle.

Sol :



Since XY || BC, and AB meets them, therefore

$\angle BXY + \angle XBC = 2$ rt.∠s. ....(1)

Also, since AB = AC, ∠B = ∠C .....(2)

From (1) and (2), we find that ,

∠BXY + ∠BCY = 2rt. ∠s.

Since a pair of opposite angles of the quadrilateral BCYX is supplementary, therefore it is cyclic, i.e., the points, B, C, X, Y lie on a circle

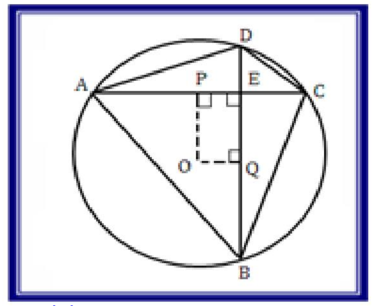
[6 marks]

20. ABCD is a cyclic quadrilateral with $AC \perp BD$. AC meets BD at E. Prove that

$EA^2 + EB^2 + EC^2 + ED^2 = 4R^2$.

Sol :

Let O be the centre of the circle, and P, Q be the feet of perpendiculars from O on AC and BD respectively. Then



$AE^2 + EC^2 = (AP + PE)^2 + (PC - PE)^2$,

$= 2AP^2 + 2PE^2$, since AP = PC

$BE^2 + ED^2 = (BQ + QE)^2 + (QD - QE)^2$

$= 2BQ^2 + 2QE^2$, since BQ = QD.

$\therefore \quad AE^2 + BE^2 + CE^2 + DE^2$

$= 2\ (AP^2 + PE^2 + BQ^2 + QE^2)$,

$= 2\ (AP^2 + PO^2 + BQ^2 + QO^2)$,

since QE = PO, PE = QO

$= 2\ (AO^2 + BO^2) = 4R^2$ [6 marks]