FEW OF THE GEMS

1. The sum of the squares of the lengths of the diagonals of a parallelogram is equal to the sum of the squares of the lengths of its sides.

2. Given two triangles having one vertex A in common, the other vertices being situated on two straight lines passing through A then ratio of the areas of these triangles is equal to the ratio of the products of the two sides of each triangle emanating from the vertex A.

3. The area of the circumscribed polygon is equal to "rp", where 'r' is the radius of the inscribed circle and 'p' its half-perimeter (in particular, this formula holds true for a triangle).

4. The radius of the circle inscribed in a right triangle can be computed by the formula $r = \frac{a+b-c}{2}$, where a and b are the legs and c is the hypotenuse.

5. If a and b are two sides of a triangle, α the angle between them, and 'I' the bisector of this angle, then $I = \frac{2ab\cos\frac{\alpha}{2}}{a+b}$,

6. Prove that the distances from the vertex A of the triangle ABC to the points of tangency of the inscribed circle with the sides AB and AC are equal to p - a (each), where p is the half-perimeter of the triangle ABC, a = | BC |.

7. The legs of a right triangle are a and b. Find the distance from the vertex of the right angle to the nearest point of the inscribed circle.

8. Given in a triangle ABC are three sides: $|BC| = a$, $|CA| = b$, $|AB| = c$. Find the ratio in which the point of intersection of the angle bisectors divides the bisector of the angle B.

9. The sum of distances from any point inside an equilateral triangle to its sides is equal to the altitude of this triangle.

10. Find the area of the quadrilateral bounded by the angle bisectors of a parallelogram with sides a and b and angle a.

11. Prove that the bisector of the right angle in a right triangle bisects the angle between the median and the altitude drawn to the hypotenuse.

12. In a triangle ABC, the angle ABC is a. Find the angle AOC, where O is the centre of the inscribed circle.

13. A circle is circumscribed about an equilateral triangle ABC, and an arbitrary point M is taken on the arc BC. Prove that $|AM| = |BM| + |CM|$. *See the figure.13.*

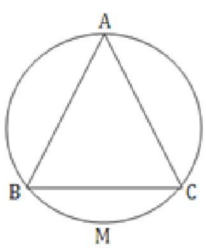


Fig. 13

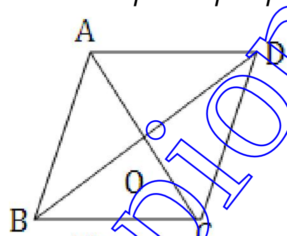


Fig. 14

14. The area of a rhombus is equal to S, the sum of its diagonals is m. Find the side of the rhombus. *See the figure.14.*

15. A square with side a is inscribed in a circle. Find the side of the square inscribed in one of the segments thus obtained.

16. A circle is circumscribed about a triangle ABC where |BC| = a, ∠B = a, ∠C = β. The bisector of the angle A meets the circle at a point K. Find |AK|.

17. Find the sum of the squares of the distances from the point M taken on a diameter of a circle to the end points of any chord parallel to this diameter if the radius of the circle is R, and the distance from M to the centre of the circle is a.

PIONEER'S SHORTCUTS

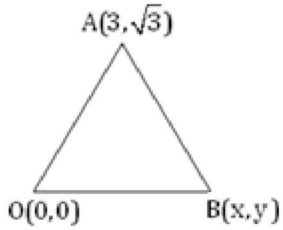
SHORTCUT NO. 1

A If two vertices of an equilateral triangle are $(x_1 y_1)$ and (x_2, y_2) then co-ordinates of the third vertex are:

$$\left(\frac{x_1 + x_2 \pm \sqrt{3}(y_2 - y_1)}{2}, \frac{y_1 + y_2 \pm \sqrt{3}(x_2 - x_1)}{2}\right)$$

Illustration: An equilateral triangle has one vertex at the point (0, 0) and another at $(3, \sqrt{3})$. Find the co-ordinates of the third vertex

Solution: Let O = (0, 0) and A = $(3, \sqrt{3})$ be the given points and let B = (x, y) be the required point. Then



$OA=OB=AB \quad \Rightarrow (OA)^2 = (OB)^2 = (AB)^2$

$\Rightarrow (3-0)^2 + (\sqrt{3}-0)^2 = (x-0)^2 + (y-0)^2 = (x-3)^2 + (y-\sqrt{3})^2$

$\Rightarrow 12 = x^2 + y^2 = x^2 + y^2 - 6x - 2\sqrt{3}y + 12$

Taking first two members then $x^2+y^2=12$ ....(1)

and taking last two members then

$6x + 2\sqrt{3}y = 12$ or $y = \sqrt{3}(2-x)$ ....(2)

(2) From (1) and (2), we get

$x^2 + 3(2-x)^2 = 12$ or $4x^2 - 12x = 0 \Rightarrow x = 0, 3$

Putting x = 0, 3 in (2), we get $y = 2\sqrt{3} - \sqrt{3}$

Hence, the co-ordinates of the third vertex B are $(0, 2\sqrt{3})$ or $(3, -\sqrt{3})$.

Pioneer Smart Solution

According to important note:

$$\left(\frac{x_1 + x_2 \pm \sqrt{3}(y_2 - y_1)}{2}, \frac{y_1 + y_2 \pm \sqrt{3}(x_2 - x_1)}{2}\right)$$

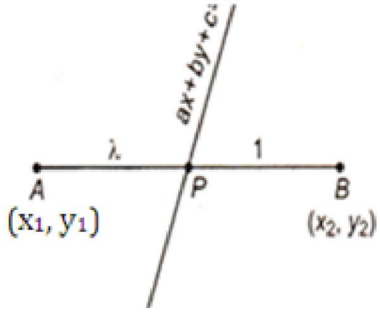
i, e., $\left(\frac{0+3 \pm \sqrt{3}(\sqrt{3}-0)}{2}, \frac{0+\sqrt{3} \pm \sqrt{3}(3-0)}{2}\right)$

or $\left(\frac{3 \pm 3}{2}, \frac{\sqrt{3} \pm 3\sqrt{3}}{2}\right)$

$\Rightarrow (0, 2\sqrt{3})$ or $(3, -\sqrt{3})$

SHORTCUT NO. 2

The straight line ax + by + c = 0 divides the joint of points A $(x_1\ y_1)$ and B (x_2,y_2) in the ratio



$$\frac{AP}{PB}=\frac{\lambda}{1}.=-\frac{(ax_1+by_1+c)}{(ax_2+by_2+c)}$$

Illustration:- Determine the ratio in which y – x +2 =0 divided the line joining (3-1) and (8,9).

Solutions:

Suppose the line $y-x+2=0$ divided the line segment joining A (3,-1) and B (8,9) in the ratio $\lambda:1$ at the point P, then the coordinate of the point P are

$\left(\frac{8\lambda+3}{\lambda+1},\frac{9\lambda-1}{\lambda+1}\right)$. But P lies on $y-x+2=0$

therefore

$$\left(\frac{9\lambda-1}{\lambda+1}\right)-\left(\frac{8\lambda+3}{\lambda+1}\right)+2=0$$

$$\Rightarrow\quad 9\lambda-1-8\lambda-3+2\lambda=0$$

$$\Rightarrow\quad 3\lambda-2=0\ \text{or}\ \lambda=\frac{2}{3}$$

So, the required ratio is $\frac{2}{3}:1$, i.e, 2:3 (internally) since here λ is positive.

Pioneer Smart Solution:

According to Note4:

$$\lambda=-\left(\frac{-1-3+2}{9-8+2}\right)=\frac{2}{3}$$

or $\lambda:1=2:3$

If ratio is positive then divides internally and if ratio is negative then divides externally.

SUB-SHORTCUT NO. 1

If mid points of the sides of a triangle are (x_1,y_1), (x_2, x_2) and (x_3, x_3) then coordinates of the original triangle are

$(x_2+x_3-x_1, y_2+y_3-y_1), (x_3+x_1-x_2, y_3+y_1-y_2)$

and $(x_1+x_2-x_3, y_1+y_2-y_3)$. $(3\alpha-x_1-x_2, 3\beta-y_1-y_2)$

SUB-SHORTCUT NO. 2: If two vertices of a triangle are (x_1,y_1) and (x_2, y_2) and the co-ordinates of centroid are (a, β) then co-ordinates of the third vertex are: $(3\alpha-x_1-x_2,\ 3\beta-y_1-y_2)$

SUB-SHORTCUT NO. 3: The orthocenter, the nine point centre the centroid and the circum center therefore all lie on a straight line.

SUB-SHORTCUT NO. 4: If O is orthocenter, N is nine point centre, G is centroid and C is circum center then to remember it see **ONGC** (i.e., Oil Natural Gas Corporation) in left of G are 2 and in right is 1 therefore G divides 0 and C in the ratio 2 : 1 (internally).

SUB-SHORTCUT NO. 5: N is the mid point of O and C

SUB-SHORTCUT NO. 6 Radius of nine point circle = $\frac{1}{2}\times$ Radius of circumcircle

Note: 1. The distance between the orthocenter and circumcenter in an equilateral triangle is zero.

2. The orthocenter of a triangle having vertices (α,β), (β,α) and (α, α) is (α, α).

3. If the orthocenter and centroid of a triangle are respectively (α, β), (γ, δ) then orthocenter will be (3γ−2α, 3δ−2β).

SUB-SHORTCUT NO. 7 If $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$ are the sides of a triangle then the area of the triangle is given by (without solving the vertices)

$$\Delta = \frac{1}{2|C_1C_2C_3|}\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2$$

Where C_1 C_2, C_3 are the cofactors of c_1, c_2, c_3 in the determinant

Here, $C_1 = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)$ $C_2 = \begin{vmatrix} a_3 & b_3 \\ a_1 & b_1 \end{vmatrix} = (a_3b_1 - a_1b_3)$ and $C_3 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = (a_1b_2 - a_2b_1)$

Illustration:- Find the area of the triangle formed by the straight lines 7x – 2y + 10 =0, 7x + 2y – 10 = 0 and 9x +y + 2 = 0 (without solving the vertices of the triangle).

Solution: The given lines are:
$$7x - 2y + 10 = 0$$
$$7x + 2y - 10 = 0$$
$$9x + y + 2 = 0$$

∴ Area of triangle $\Delta = \frac{1}{2|C_1C_2C_3|}\begin{vmatrix} 7 & -2 & 10 \\ 7 & 2 & -10 \\ 9 & 1 & 2 \end{vmatrix}^2$ ....(1)

where $C_1 = \begin{vmatrix} 7 & 2 \\ 9 & 1 \end{vmatrix} = 7 - 18 = -11,$ $C_2 = \begin{vmatrix} 9 & 1 \\ 7 & -2 \end{vmatrix} = -18 - 7 = -25$

and $C_3 = \begin{vmatrix} 7 & -2 \\ 7 & 2 \end{vmatrix} = 14 + 14 = 28,$ and $\begin{vmatrix} 7 & -2 & 10 \\ 7 & 2 & -10 \\ 9 & 1 & 2 \end{vmatrix} = 10C_1 - 10C_2 + 2C_3$

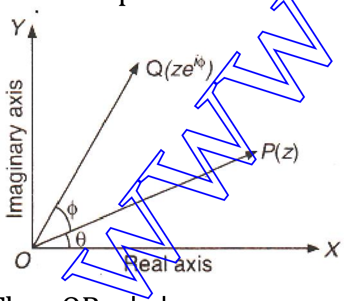
$10\times(-11) - 10\times(-25) + 2\times 28 = 196$

∴ From (1), $\Delta = \frac{1}{2|-11\times(-25)\times 28|}\times(196)^2 = \frac{196\times 196}{2\times 11\times 25\times 28} = \frac{686}{275}$ sq units

Complex number as a rotating arrow in Argand plane:

Let $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$ ... (1)

be a complex number representing a point P in the Argand plane.



Then OP = |z| = r

and ∠POX = θ

Now consider complex number $z_r = ze^{i\phi}$

or $z_1 = re^{i\theta}\ e^{i\phi} = r.e^{i(\theta+\phi)}$ {from (1)}

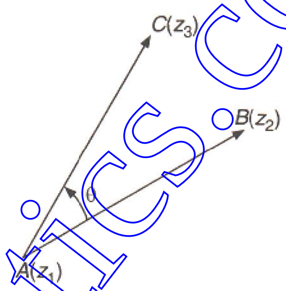
Clearly the complex number Z_1 represents a point Q in the Argand plane, when

$OQ = r$ and $\angle QOX = \theta + \phi$

Clearly multiplication of z with $e^{i\phi}$ rotates the vector $\overrightarrow{OP}$ through angle (j> in anticlockwise sense.

Similarly multiplication of z with $e^{-i\phi}$ will rotate the vector $\overrightarrow{OP}$ in clockwise sense.

Note: If z_1, z_2 and z_3 are the affixes of the three points A, B and C such that AC = AB and $\angle CAB = \theta$ Therefore



$\overrightarrow{AB} = z_2 - z_1, \overrightarrow{AC} = z_3 - z_1$

Then $\overrightarrow{AC}$ will be obtained by rotating $\overrightarrow{AB}$ through an angle θ in anticlockwise sense and therefore $\overrightarrow{AC} = \overrightarrow{AB}e^{i\theta}$

Or $(z_3 - z_1) = (z_2 - z_1)e^{i\theta}$

or $\left(\dfrac{z_3 - z_1}{z_2 - z_1}\right) = e^{i\theta}$

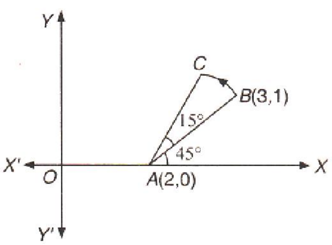
or $(z_3 - z_1) = (z_2 - Z_1)e'^e$

ILLUSTRATION: The line joining the-points A (2, 0) and B (3,1) is rotated about A in the anticlockwise direction through an angle of 15°. Find the equation of the line in the new position. If B goes to C in the new position, what will be the co-ordinates of C?

Solution: Here AB = $\sqrt{(2-3)^2 + (0-1)^2} = \sqrt{2}$

and slope of $AB = \dfrac{1-0}{3-2}$ = 1 = tan 45° 3-2

$\angle BAX = 45°$



Now line AB is rotated through an angle of 15°

$\Rightarrow \quad \angle CAX = 45° + 15° = 60°$ and $AB = AC = \sqrt{2}$

Equation of line AC in parametric form is

$\left.\begin{array}{l} x = 2 + r\cos 60° \\ y = 0 + r\sin 60° \end{array}\right\}$ Since $AC = r = \sqrt{2}$

Put $r = \sqrt{2}$ in (1), then $x = 2 + \sqrt{2}.\dfrac{1}{2} = \dfrac{4+\sqrt{2}}{2}$ and $y = \sqrt{2}.\dfrac{\sqrt{3}}{2} = \dfrac{\sqrt{6}}{2}$

Equation of the line AC is $\dfrac{x+2}{y} = \cot 60^0 = \dfrac{1}{\sqrt{3}}$ or $x\sqrt{3} - y + 2\sqrt{3} = 0$

and co-ordinates of C are $\left(\dfrac{4+\sqrt{2}}{2}, \dfrac{\sqrt{6}}{2}\right)$.

Alternative Method:

$\because$ A = (2,0), B = (3,1), let C = (x,y)

∴ $z_A = 2$, $z_B = 3 + i$, $z_c = x + iy$ = $\frac{z_C - z_A}{z_B - z_A} = e^{i\frac{5\pi}{12}}$

$z_c = 2 = (1 + i(\cos 15^0 + i\sin 15°)$ or

$$= z_C = 2 + (1+i)\left(\frac{\sqrt{3}+1}{2\sqrt{2}} + i\frac{\sqrt{3}+1}{2\sqrt{2}}\right))$$

$$= \left(2\frac{\sqrt{3}+1}{2\sqrt{2}} - \frac{\sqrt{3}-1}{2\sqrt{2}}\right) + i\left(\frac{\sqrt{3}+1}{2\sqrt{2}} + \frac{\sqrt{3}-1}{2\sqrt{2}}\right)$$

$$= \left(2 + \frac{1}{\sqrt{2}}\right) + i\left(\frac{\sqrt{3}}{\sqrt{2}}\right)$$

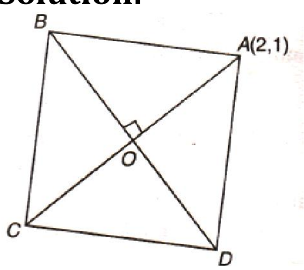
$$= \frac{4+\sqrt{2}}{2} + i\left(\frac{\sqrt{6}}{2}\right)$$

$$\therefore\ C = \left(\frac{4+\sqrt{2}}{2}, \frac{\sqrt{6}}{2}\right)$$

and equation of AC, y − 0 = tan 60° (x −2), $x\sqrt{3} - y - 2\sqrt{3} = 0$

ILLUSTRATION: The centre of a square is at the origin and one vertex is A (2,1). Find the co-ordinates of oilier vertices of the square.

Solution:



∵ A = (2,1) ∴ $z_A = 2+i$ Now in triangle AOB, OA=OB, ∠AOB = 90° = $\frac{\pi}{2}$

∴ $Z_B = Z_A e^{i\frac{\pi}{2}} = iz_A = 2i - 1$

∵ $B = (-1, 2)$

∵ O is the mid point of AC and BD

∴ $C = (-2, -1)$ and $D = (1, -2)$

ILLUSTRATION: The extremities of the diagonal of a square are (1,1), (-2, -1). Obtain the other two vertices and the equation of the other diagonal.

Solution:

$\because$ A = (1,1) $\therefore$ $Z_A = 1+i$

and C = (−2,−1) $z_c = -2 - i$

then centre of $E = \left(-\frac{1}{2}, 0\right)$ $\therefore$ $z_E = -\frac{1}{2}$

Now in △AEB, (EA = EB) $\frac{z_B - z_E}{z_A - z_E} = e^{-\frac{\pi}{2}} = i$

$\Rightarrow z_B = -\frac{3}{2} + \frac{3}{2}i$ then $D = \left(-1 + \frac{3}{2}, -\frac{3}{2}\right)$

$B = \left(-\frac{3}{2}, \frac{3}{2}\right)$, $D = \left(\frac{1}{2}, -\frac{3}{2}\right)$

Hence equation of other diagonal BD is

$$y - 0 = \frac{\frac{3}{2} - 0}{-\frac{3}{2} + \frac{1}{2}}\left(x + \frac{1}{2}\right) \Rightarrow 6x + 4y + 3 = 0$$

Let ABCD be a square

and let A (-1,-2) and C (3,2) be the given points.

Let B (x,y) be the unknown vertices

$\therefore$ AB = BC

$\Rightarrow AB^2 = BC^2$

$(x+1)^2 + (y-2)^2 = (x-3)^2 + (y-2)^2$

x = 1.......................(i)

In right angled triangle, ABC,

we have $(AB^2 + BC^2 = (AC)^2$

$2x^2 + 2y^2 - 4x - 8y + 18$

$= (3+1)^2 + (2-2)^2$

$y^2 - 4y = 0$

y = 0 , y = 4

Hence required vertices of square are (1,0) and (1,4)

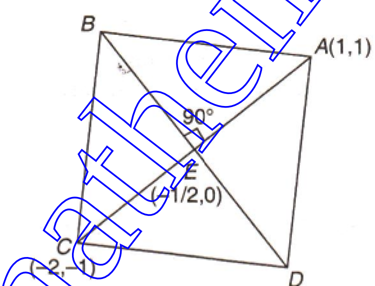
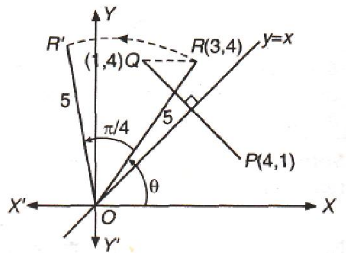


ILLUSTRATION: The point (4,1) undergoes the following three transformations successively :
(i) Refection about the line y = x .
(ii) Translation through distance 2 units along the positive direction of x-axis.
(iii) Rotation through an angle π/ 4abnui the origin in the anticlockwise direction. Then find the co-ordinates of the final position.



Solution: Let Q (x_1, y_1) be the reflection of P about the line y = x. Then

$$\left.\begin{aligned} x_1 &= 1 \\ y_1 &= 1 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots(1)$$

Co-ordinates of Q is (1, 4).
Given that Q move 2 units along the positive direction of x-axis.
∴ Co-ordinates of R is (x_1 + 2, y_1) or R(3, 4)
If OR makes an angle θ, then

$$\tan\theta = \frac{4}{3} \quad \therefore \quad \sin\theta = \frac{4}{3} \text{ and } \cos\theta = \frac{3}{5}$$

After rotation of $\frac{\pi}{4}$ let new position of R is R' and

4 OR=OR' = $\sqrt{3^2 + 4^2}$ =5
∴ OR' makes an angle (π/ 4 + θ)with x-axis.
Co-ordinates of R'

$$\left(OR'\cos\left(\frac{\pi}{4}+\theta\right), OR'\sin\left(\frac{\pi}{4}+\theta\right)\right)$$

$$\text{i,e.,} \quad R'\begin{pmatrix} OR'\left(\frac{1}{\sqrt{2}}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta\right), \\ OR'\sin\left(\frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta\right) \end{pmatrix}$$

$$\Rightarrow \quad R'\left(5\left(\frac{3}{5\sqrt{2}} - \frac{4}{5\sqrt{2}}\right), 5\left(\frac{3}{5\sqrt{2}} + \frac{4}{5\sqrt{2}}\right)\right)$$

$$\Rightarrow \quad R'\left(-\frac{1}{\sqrt{2}}, \frac{7}{\sqrt{2}}\right)$$

Pioneer Smart Solution (Use of complex number)

Let Q be the reflection of P (4,1) about the line y = x, then Q = (1, 4)
∵ Q move 2 units along the +ve direction of x-axis, if new point is R then R = (3, 4).
If R(3,4) = R (z_1)
when z_1 = (3 + 4i)
then R'(x, y) = R'(z_2)

$$z_2 = z_1 e^{i\pi/4} \qquad \left(\because \angle ROR' = \frac{\pi}{4}\right)$$

$$= (3+4i)\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \quad = (3+4i)\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$$

$$= \left(-\frac{1}{\sqrt{2}} + \frac{7i}{\sqrt{2}}\right)$$

Hence new co-ordinates are $\left(-\frac{1}{\sqrt{2}}, \frac{7}{\sqrt{2}}\right)$.

"OBJECTIVE QUESTIONS"

1. $(1+i)^6+(1-i)^6=$

(a) 2^8 (b) 0 (c) −1 (d) 1

Sol. (D) Alternative Method I:

Binomial coefficients in $(1+x)^6$ are

${}^6C_0, {}^6C_1, {}^6C_2, {}^6C_3, {}^6C_4, {}^6C_5, {}^6C_6$

$(1+i)^6 = +++++++$..........................(i)

$(1-i)^6 = +-+-+-+$..........................(ii)

adding (i)and (ii), the terms those exist at even places get cancelled

$\therefore\ (1+i)^6+(1-i)^6 = 2[{}^6C_0+{}^6C_2 i^2+{}^6C_4 i^4+{}^6C_6 i^6]$

$= 2[{}^6C_0-{}^6C_2+{}^6C_4-{}^6C_6]$ using $i^2=-1$

$=2(0)\qquad {}^6C_2={}^6C_4\qquad =0$

Alternative method II:

$1+i=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$

$1+i=\sqrt{2}\ \text{cis}\frac{\pi}{4}$

$1+i=\sqrt{2}\ \text{cis}\left(-\frac{\pi}{4}\right)$

$(\sqrt{2})^6\left[\text{cis}\frac{3\pi}{2}+\text{cis}\left(-\frac{3\pi}{2}\right)\right]$, $(\sqrt{2})^6\left[2\cos\frac{3\pi}{2}\right]=0$

"Pioneer Smart Solution":

$z=-1=\cos\pi+i\sin\pi$

$z^{2/3}=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$

$z^{2/3}=\cos\left(\frac{2k\pi+2\pi}{3}\right)+i\sin\left(\frac{2k\pi+2\pi}{3}\right)$

$z^{2/3}=\text{cis}\frac{2\pi}{3}, \text{cis}\frac{4\pi}{3}, \text{cis}\ 2\pi$

$=\text{cis}\frac{2\pi}{3}, \text{cis}\left(2\pi-\frac{2\pi}{3}\right), \text{cis}\ 2\pi$

$=\text{cis}\frac{2\pi}{3}, \text{cis}\left(-\frac{2\pi}{3}\right), \text{cis}\ 0$

$\text{Arg}\ z^{2/3}=\frac{2\pi}{3}, -\frac{2\pi}{3}, 0$

2. The square root of −5−12i is

(a) $\pm(3+2i)$ (b) $(2+3i)$ (c) $\pm(2-3i)$ (d) $\pm(3-2i)$

Sol. (c)

$-5-12i=-5-2(6i)$

$=-5-2\left(\sqrt{-36}\right)\quad =-5-2\sqrt{(-9)(4)}$

$-5-2\sqrt{(3i)^2(2)^2}$

$=-9+4-2.2(3i)\quad =(-3i)^2+2^2+2.2(-3i)$

$-5-12=(2-3i)^2$

$\therefore\ \sqrt{-5-12i}=\pm(2-3i)$

"Pioneer Smart Solution".

Every complex number possesses its two square roots. From these two both, one or none may or may not be in the choice, so be careful about it.

Fact

$\sqrt{a+ib}=\pm\left[\sqrt{\frac{|z|+\text{Re}z}{2}}-i\sqrt{\frac{|z|-\text{Re}z}{2}}\right]$ a is $b<0$

$\sqrt{-5-12i}\left[\sqrt{\frac{13+(-5)}{2}}-i\sqrt{\frac{13-(-5)}{2}}\right]=\pm(2-3i)$

3. If $z=\left(\frac{\sqrt{3}+i}{2}\right)^6+\left(\frac{\sqrt{3}-i}{2}\right)^6$ then

(a) Re z = 0 (b) Re z, Im z > 0 (c) Im (z) = 0 (d) Re z > 0, Im (z) < 0

Sol. (c)

$$z=\left(\frac{\sqrt{3}+i}{2}\right)^6+\left(\frac{\sqrt{3}-i}{2}\right)^6$$

Fact: Using fact every complex number a +ib for which $|a:b|=1:\sqrt{3}$ or $\sqrt{3}:1$ can be expressed in terms of i,ω,ω^2.

$$\therefore \left(\frac{\sqrt{3}+i}{2}\right)^6+\left(\frac{\sqrt{3}-i}{2}\right)^6=\left(\frac{-1+i\sqrt{3}}{2}\right)^6-\left(\frac{-1-i\sqrt{3}}{2}\right)^6$$

$$=\left(\frac{\omega}{i}\right)^6-(-\omega^2)^6 \qquad =\frac{\omega^6}{i}-\omega^{12}=-2=-2+0i$$

$$\therefore \ \text{Im}(z)=0$$

"Pioneer Smart Solution":

$$z=\left(\frac{\sqrt{3}+1}{2}\right)^6+\left(\frac{\sqrt{3}-i}{2}\right)^6$$

$$=e^{i\pi}+e^{-i\pi} \qquad =2\cos\pi$$

$$z=-2 \quad =-2+0i$$

4. If $\alpha=\cos\frac{8\pi}{11}+i\sin\frac{8\pi}{11}$ then $\text{Re}(\alpha+\alpha^2+\alpha^3+\alpha^4+\alpha^5)$ equals

(a) 0 (b) −1/2 (c) 1/2 (d) None of these

Sol. (b)

Def: $R_e z=\frac{z+\bar{z}}{2}$

$$\therefore R_e(\alpha+\alpha^2+\alpha^3+\alpha^4+\alpha^5)=\frac{(\alpha+\alpha^2+\alpha^3+\alpha^4+\alpha^5}{}$$

$$=\frac{1}{2}\left(\alpha+\alpha^2+\alpha^3+\alpha^4+\alpha^5+\frac{1}{\alpha^2}+\frac{1}{\alpha^3}+\frac{1}{\alpha^4}+\frac{1}{\alpha^5}\right)$$

$$\frac{1}{2\alpha^2}[\alpha^6+\alpha^7+\alpha^8+\alpha^9+\alpha^{10}+\alpha^4+\alpha^3+\alpha^2+\alpha+1]$$

$$\frac{1}{2\alpha^2}[1+\alpha+.........+\alpha^{10}-\alpha^5]$$

$$\frac{1}{2}\frac{1}{\alpha^2}\left[\frac{1-\alpha^{11}}{1-\alpha}-\alpha^5\right]\ \frac{1}{2\alpha^2}[0-\alpha^5]=-\frac{1}{2}$$

"Pioneer Smart Solution":

Given $\alpha=\cos\frac{8\pi}{11}+i\sin\frac{8\pi}{11}$

$$\therefore \ \alpha^{11}=\cos\ 8\pi+i\sin\ 8\pi=1$$

$$\text{so}\sum_{n=0}^{10}\alpha^n=\frac{1-\alpha^{11}}{1-\alpha}=0 \quad (\text{sum of } 11^{th} \text{ roots of units})$$

$$\text{now } \text{Re}(z)=\frac{z+\bar{z}}{2}=\frac{\text{sum of } 11^{th}\text{ root of units}-1}{2}=-\frac{1}{2}$$

5. If $z = 1 + i\sqrt{3}$ then z^6 equals

(a) 32 (b) −32 (c) 64 (d) None of these

Sol. (c)

$z = 1 + i\sqrt{3}$

$z^6 = \left(1 + i\sqrt{3}\right)^6$

$= {}^6C_0 + {}^6C_1\left(i\sqrt{3}\right) + {}^6C_1\left(i\sqrt{3}\right)^2 \, {}^6C_3\left(i\sqrt{3}\right)^3 + {}^6C_3\left(i\sqrt{3}\right)^4$

$= ({}^6C_0 - {}^6C_2(3) + {}^6C_4 9 - {}^6C_0 27) + i\sqrt{3}({}^6C_1 - 3\,{}^6C_3 + 9^6$

$= (1 = -15 \times 3 + 135 - 27) + i\sqrt{3}(60 - 60)$

$= (136 - 72) + i\sqrt{3}(0) = 64$

"Pioneer Smart Solution (i)"

$Z = 1 + i\sqrt{3}$

$= -\left(\frac{-1 - i\sqrt{3}}{2}\right) \times 2 \quad = -2\omega^2$

$z^6 = (-2)^6(\omega^2)^6 \quad = 2^6\omega^{12}$

$\therefore \; z^6 = 2^6 = 64$

"Pioneer Smart Solution (ii)"

$z = 1 + i\sqrt{3}$

$z = 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

$z = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$z = 2e^{i\pi/3}$

$\therefore z^6 = 2^6 \; e^{2i\pi} = 2^6$

6. The complex number $z = x + iy$ which satisfying the equation $\left|\frac{z - 7i}{z + 7i}\right| = 1$

(a) x – axis (b) y – axis (c) on a circle (d) the line y = 7

Sol. (a)

Given $\left|\frac{z - 7i}{z + 7i}\right| = 1$

$\Rightarrow |z - 7i| = |z + 7i|$

$\Rightarrow |x + i(y - 7)| = |x + i|(y - 7)|$

$\Rightarrow x^2 + (x - 7)^2 = x^2 + (y - 7)^2$ (after taking absolute value squaring both side)

$\Rightarrow (y - 7)^2 - (y - 7)^2 = 0$

$\Rightarrow \quad 28y = 0$

$y = 0$ (Equation of x-axis)

$\Longrightarrow$ z lies on x – axis.

"Pioneer Smart Solution":

Given $\left|\frac{z - 7i}{z + 7i}\right| = 1$

$\Longrightarrow$ z lies on the right bisector of the line segment connection the points 7i, −7i.

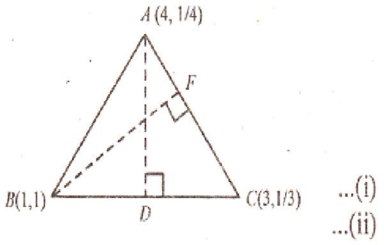
$\therefore$ z lies on x-axis

Hence z lies on real axis.

7. If the vertices of a triangle is (4,1/4)(3,1/3),(1,1) then orthocenter of the triangle is

(a) $\left(\frac{1}{12}, 12\right)$ (b) $\left(12, \frac{1}{12},\right)$ (c) $\left(-\frac{1}{12}, -12\right)$ (d) None of these

Sol. (c): Orthocenter is point of intersection of altitudes drawn from one vertex to opposite side. In order to determine the co-ordinate of orthocenter we need the equations of altitudes



Slope of BC $\frac{\frac{1}{3}-1}{3-1} = -1/3$

$\therefore$ Equation of AD is $y - 1/4 = 3(x - 4)$

$12x - 4y = 47$

Equation of BF is $12x - y = 11$

By solving (i) and (ii) we get the co-ordinate of orthocenter

$\therefore \quad (x, y) = \left(-\frac{1}{12}, 12\right)$

"Pioneer Smart Solution":

Using Fact: If vertices of a ΔABC are (a,1/a), (b,1/b),(c,1/c) then coordinate of orthocenter is $\left(-\frac{1}{abc}, -abc\right)$

$\therefore$coordinate of orthocenter

$= \left(-\frac{1}{(4)(3)(2)}, -4(1)(3)\right) = \left(-\frac{1}{12}, -12\right)$

8. The angle between the pair of tangents drawn from the point (2,4) to the circle $x^2 + y^2 = 4$ is

(a) $\tan^{-1}\left(\frac{3}{8}\right)$ (b) $\tan^{-1}\left(\frac{4}{3}\right)$ (c) 90^0 (d) None of these

Sol. (b): Equation of pair of tangent to a circle is

$T^2 = SS_1$ where $T = xx_1 + yy_1 - 4$

$S_1 = x_1^2 + y_1^2 - 4$

$S = x^2 + y^2 - 4$

$\therefore (2x + 4y - 4)^2 = (x^2 + y^2 - 4)(4 + 16 - 4)$

$4(x + 2y - 2)^2 = 16(x^2 + y^2 - 4)$

$x^2 + 4y^2 + 4xy - 8y - 4x + 4 = 4x^2 + 4y^2 - 16$

$3x^2 - 4xy + 8y - 20 = 0 =$

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c$

$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a + b} = \frac{2 \times 2}{3}$

$\theta \quad = \tan^{-1}(4/3)$

"Pioneer Smart Solution":

The length of tangent

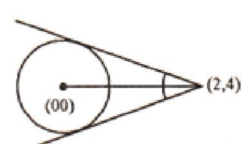
$= PT = \sqrt{x_1^2 + h_1^2 - a^2} = \sqrt{2^2 + 4^2 - 4}$

$\therefore \ PT = 4$

radius = OT = 2

now $\tan\alpha = \frac{2}{4} = \frac{1}{2}$

and $\tan\theta = \tan 2\alpha = \frac{2\tan\alpha}{1 - \tan^2\alpha} = \frac{2.\frac{1}{2}}{1 - 1/4} = \frac{4}{3}$



Θ = α+ α = α

9. The radius of the circle $3x^2+3y^2+9x+8y-4=0$ is

(a) $\frac{\sqrt{193}}{3}$ (b) $\frac{\sqrt{193}}{6}$ (c) $\frac{\sqrt{129}}{3}$ (d) None of these

Sol. (b) Method 1:

$3x^2+3y^2+9x-8y-4=0$

$\Rightarrow x^2+y^2+3x-\frac{8}{3}y-\frac{4}{3}=0$

$\Rightarrow (x^2+3x)+\left(y^2-\frac{8}{3}y\right)=\frac{4}{3}$

$\Rightarrow \left(x+\frac{3}{2}\right)^2+\left(y-\frac{4}{3}\right)^2=\frac{4}{3}+\left(\frac{4}{3}\right)^2+\left(\frac{3}{2}\right)^2$

$=\frac{4}{3}+\frac{16}{9}+\frac{9}{4}=\frac{48+64+81}{36}=\frac{193}{36}$

$\left(x+\frac{3}{2}\right)^2\left(y-\frac{4}{3}\right)^2=\left(\frac{\sqrt{193}}{6}\right)^2=(x-x_1)^2(y-y_1)^2=r^2$

Method 2:

$3x^2+3y^2+9x-8y-4=0$

$\Rightarrow x^2+y^2+3x-\frac{8}{3}y-\frac{4}{3}=0$

$\Rightarrow x^2+y^2+2\left(\frac{3}{2}\right)x+2\left(-\frac{4}{3}\right)-\frac{4}{3}=0=x^2+y^2+2fy+c$

$\Rightarrow$ centre $(-g,-f)$, $R=\sqrt{g^2+f^2-c}$

centre $\left(-\frac{3}{2},\frac{4}{3}\right)$, $c=-4/3$

"Pioneer Smart Solution":
Fact: If $ax^2+ay^2+2gx+2fy+c=0$ be equation of circle then

$r=\frac{\sqrt{g^2+f^2-ac}}{a}$

$\therefore 3x^2+3y^2+9x-8y-4=0=3x^2+2\left(\frac{9}{2}\right)x-2(4)x-4=0$

$\Rightarrow R=\sqrt{\frac{\left(\frac{9}{2}\right)^2+(4)^2-(3)(-4)}{3}}$

$R=\sqrt{\frac{\frac{81}{4}+16+12}{3}}=\sqrt{\frac{81+28\times4}{6}}=\sqrt{\frac{193}{6}}$

"SPECIAL CONSTANTS"

1. π = 3.14159 26535 89793 23846 2643 .....

2. e = 2.71828 18284 59045 23536 0287 ..... = $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ natural base of logarithms.

3. $\sqrt{2}$ = 1.41421 35623 73095 0488 ......

4. $\sqrt{3}$ = 1.73205 08075 68877 2935 ......

5. $\sqrt{5}$ = 2.23606 79774 99789 6964 ......

6. $\sqrt[3]{2}$ = 1.25992 1050 ......

7. $\sqrt[3]{3}$ = 1.44224 9570 ......

8. $\sqrt[5]{2}$ = 1.14869 8355 ......

9. $\sqrt[5]{3}$ = 1.24573 0940 ......

10. $e^{\pi} = 23.14069\ 26327\ 79269\ 006$ ......

11. $\pi^{e} = 22.45915\ 77183\ 61045\ 47342\ 715$ ......

12. $e^{e} = 15.15426\ 22414\ 79241\ 90$ ......

13. $\log_{10} 2 = 0.30102\ 99956\ 63981\ 19521\ 37389$ ......

14. $\log_{10} 3 = 0.47712\ 12547\ 19662\ 43729\ 50279$ ......

15. $\log_{10} e = 0.43429\ 44819\ 03251\ 82765$ ......

16. $\log_{10} \pi = 0.49714\ 98726\ 94133\ 85435\ 12683$ ......

17. $\log_e 10 = \ln 10 = 2.30258\ 50929\ 94045\ 68401\ 7991$ ......

18. $\log_e 2 = \ln 2 = 0.69314\ 71805\ 59945\ 30941\ 7232$ ......

19. $\log_e 3 = \ln 3 = 1.09861\ 22866\ 68109\ 69139\ 5245$ ......

20. $\gamma = 0.57721\ 56649\ 01532\ 86060\ 6512$ ...... = Euler's constant =

$$\lim_{x\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{n}-\ln n\right)$$

21. $e^{\gamma} = 1.78107\ 24179\ 90197\ 9852$ ......

22. $\sqrt{e} = 1.64872\ 12707\ 00128\ 1468$ ......

23. $\sqrt{\pi} = 1\left(\frac{1}{2}\right) = 1.77245\ 38509\ 05516\ 02729\ 8167$ ..... Where Γ is the gamma function.

24. $\Gamma\left(\frac{1}{3}\right) = 2.67893\ 85347\ 07748$ ......

25. $\Gamma\left(\frac{1}{4}\right) = 3.62560\ 99082\ 21908$ ......

26. 1 radian = $180^{o}/\pi = 57.29577\ 95130\ 8232$ ....^{o}

27. $1^{o} = \pi/180$ radians = $0.01745\ 32925\ 19943\ 29576\ 92$ .... radians.

GREEK ALPHABET

Α α	alpha	Ν ν	nu
Β β	beta	Ξ ξ	xi
Γ γ	gamma	Ο ο	omicron
Δ δ	delta	∏ π	pi
Ε ∈ ε	epsilon	Ρ ρ	rho
Ζ ζ	zeta	∑ σ	sigma
Η η	eta	Τ τ	tau
Θ θ,	theta	υ	upsilon
Ι ι	iota	Φ ∅, φ	phi
Κ κ	kappa	Χ χ	chi
Λ λ	lambda	Ψ ψ	psi
Μ μ	mu	Ω ω	omega

"Special Power Series"

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^r}{r!} + \dots$ (all x)

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + \frac{(-1)^r x^{2r+1}}{(2r+1)!} + \dots$ (all x)

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + \frac{(-1)^r x^{2r}}{(2r)!} + \dots$ (all x)

$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \dots$ ($|x| < \frac{x}{2}$)

$\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\frac{x^7}{7} + \dots + \frac{1.3.5 \dots (2n-1)}{2.4.6 \dots (2n)}\frac{x^{2n+1}}{2n+1} + \dots$ ($|x| < 1$)

$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots + (-1)^n \frac{x^{2n+1}}{2n+1} + \dots$ ($|x| < 1$)

$\ell n(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + (-1)^{n+1}\frac{x^n}{n} + \dots$ ($-1 < x \le 1$)

$\text{Sinh } x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots + \frac{x^{2n+1}}{(2n+1)!} + \dots$ (all x)

$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots + \frac{x^{2n}}{(2n)!} + \dots$ (all x)

$\tanh x = x - \frac{x^3}{3!} + \frac{2x^5}{15} - \frac{17x^7}{315} + \dots$ ($|x| < \frac{x}{2}$)

$\sinh^{-1} x = x - \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} - \frac{1.3.5}{2.4.6}\frac{x^7}{7} + \dots + (-1)^n \frac{1.3.5 \dots (2n-1)}{2.4.6 \dots 2n}\frac{x^{2n+1}}{2n+1} + \dots$ ($|x| < 1$)

$\tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \dots \frac{x^{2n+1}}{2n+1} + \dots$ ($|x| < 1$)

$\frac{\pi}{2} = 1 + \frac{1}{2}\frac{1}{3} + \frac{1.3}{2.4}\cdot\frac{1}{5} + \frac{1.3.5}{2.4.6}\cdot\frac{1}{7} + \dots$

$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots\ldots\ldots$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\ldots\ldots$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots\ldots\ldots$$

$$= \frac{1}{2} + \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} \ldots\ldots\ldots\ldots$$

"Hyperbolic Identities"

$\cosh x = (e^x + e^{-x})/2$ $\sinh(x) = (e^x - e^{-x})/2$

$\tanh x = \sinh x/\cosh x$

$\text{sech} x = 1 / \cosh x$ $\text{cosech}\, x = 1/\sinh x$

$\coth x = \cosh x/ \sinh x = 1/\tanh x$

$\cosh ix = \cos x$ $\sinh ix = i \sin x$

$\cos i x = \cosh x$ $\sin ix = i \sinh x$

$\cosh^2 A - \sinh^2 A = 1$

$\text{sech}^2 A = 1 - \tanh^2 A$

$\text{cosech}^2 A = \coth^2 A - 1$

"PHYSICAL AND ASTRONOMICAL CONSTANTS"

c	Speed of light in vacuum	2.998×10^8 m s^{-1}
E	Elementary charge	1.602×10^{-19} C
m_n	Neutron rest mass	1.675×10^{-27} kg
m_p	Proton rest mass	1.673×10^{-27} kg
m_e	Electron rest mass	9.110×10^{-31} J s
h	Planck's constant	6.626×10^{-34} Js
$\hbar$	Dirac's constant (= h/2 π)	1.055×10^{34} J s
K	Boltzmann's constant	1.381×10^{-23} J K^{-1}
G	Gravitational constant	6.673×10^{-11} N m^2 kg^{-2}
σ	Stefan-Boltzmann constant	5.670×10^{-8} J m^{-2} K^{-4} s^{-1}
C_1	First Radiation Constant (= 2 π hc^2)	3.742×10^{-16} J m^2 s^{-2}
C_2	Second Radiation Constant (= hc/k)	1.439×10^{-2} m K
ε_o	Permittivity of free space	8.854×10^{-12} C^2 N^{-1} m^{-2}
μ_o	Permeability of free space	$4\pi \times 10^{-7}$ H m^{-1}
N_A	Avogadro constant	6.022×10^{23} mol^{-1}
R	Gas constant	8.314 J K^{-1} mol^{-1}

First check your Concept

- ✓ Why roots of $ax^2 + bx + c = 0$ are $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$?
- ✓ Why sum of n *terms* of an A.P. is $\frac{n}{2}[2a + (n-1)d]$?
- ✓ why $\sin 30^0 = \frac{1}{2}$?
- ✓ Why $\cos 45^0 = \frac{1}{\sqrt{2}}$?
- ✓ Why centroid of triangle whose vertices are (x_1, y_1), (x_2, y_2), (x_3, y_3) is $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$?
- ✓ Why distance between $P(x_1, y_1)$ and $Q(x_2, y_2)$ is $\sqrt{(x_2 - x_1)^2 - (y_2 - y_1)^2}$?
- ✓ Why sum of interior angles of triangle is 180^0?
- ✓ Why volume of sphere is $\frac{4}{3}\pi r^3$?
- ✓ Why curved surface area of cylinder is $2\pi rh$?
- ✓ Why area of circle is πr^2?
- ✓ Why cos is negative in second quadrant?
- ✓ Why sum of roots of quadratic equation is $-b/a$?

…………100's of many more why's.,

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