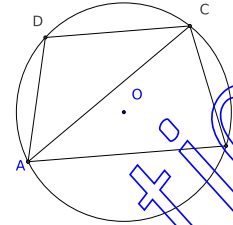
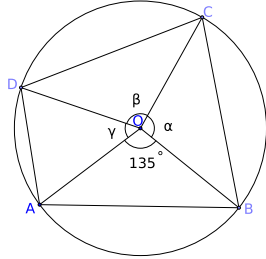


Problems and Solutions: INMO-2012

1. Let $ABCD$ be a quadrilateral inscribed in a circle. Suppose $AB = \sqrt{2 + \sqrt{2}}$ and AB subtends 135° at the centre of the circle. Find the maximum possible area of $ABCD$.



Solution: Let O be the centre of the circle in which $ABCD$ is inscribed and let R be its radius. Using cosine rule in triangle AOB , we have

$$2 + \sqrt{2} = 2R^2(1 - \cos 135^\circ) = R^2(2 + \sqrt{2}).$$

Hence $R = 1$.

Consider quadrilateral $ABCD$ as in the second figure above. Join AC . For $[ADC]$ to be maximum, it is clear that D should be the mid-point of the arc AC so that its distance from the segment AC is maximum. Hence $AD = DC$ for $[ABCD]$ to be maximum. Similarly, we conclude that $BC = CD$. Thus $BC = CD = DA$ which fixes the quadrilateral $ABCD$. Therefore each of the sides BC , CD , DA subtends equal angles at the centre O .

Let $\angle BOC = \alpha$, $\angle COD = \beta$ and $\angle DOA = \gamma$. Observe that

$$[ABCD] = [AOB] + [BOC] + [COD] + [DOA] = \frac{1}{2} \sin 135^\circ + \frac{1}{2} (\sin \alpha + \sin \beta + \sin \gamma).$$

Now $[ABCD]$ has maximum area if and only if $\alpha = \beta = \gamma = (360^\circ - 135^\circ)/3 = 75^\circ$. Thus

$$[ABCD] = \frac{1}{2} \sin 135^\circ + \frac{3}{2} \sin 75^\circ = \frac{1}{2} \left(\frac{1}{\sqrt{2}} + 3 \frac{\sqrt{3} + 1}{2\sqrt{2}} \right) = \frac{5 + 3\sqrt{3}}{4\sqrt{2}}.$$

Alternatively, we can use Jensen's inequality. Observe that α, β, γ are all less than 180° . Since $\sin x$ is concave on $(0, \pi)$, Jensen's inequality gives

$$\frac{\sin \alpha + \sin \beta + \sin \gamma}{3} \leq \sin \left(\frac{\alpha + \beta + \gamma}{3} \right) = \sin 75^\circ.$$

Hence

$$[ABCD] \leq \frac{1}{2\sqrt{2}} + \frac{3}{2} \sin 75^\circ = \frac{5 + 3\sqrt{3}}{4\sqrt{2}},$$

with equality if and only if $\alpha = \beta = \gamma = 75^\circ$.

2. Let $p_1 < p_2 < p_3 < p_4$ and $q_1 < q_2 < q_3 < q_4$ be two sets of prime numbers such that $p_4 - p_1 = 8$ and $q_4 - q_1 = 8$. Suppose $p_1 > 5$ and $q_1 > 5$. Prove that 30 divides $p_1 - q_1$.

Solution: Since $p_4 - p_1 = 8$, and no prime is even, we observe that $\{p_1, p_2, p_3, p_4\}$ is a subset of $\{p_1, p_1 + 2, p_1 + 4, p_1 + 6, p_1 + 8\}$. Moreover p_1 is larger than 3. If $p_1 \equiv 1 \pmod{3}$, then $p_1 + 2$ and $p_1 + 8$ are divisible by 3. Hence we do not get 4 primes in the set $\{p_1, p_1 + 2, p_1 + 4, p_1 + 6, p_1 + 8\}$. Thus $p_1 \equiv 2 \pmod{3}$ and $p_1 + 4$ is not a prime. We get $p_2 = p_1 + 2, p_3 = p_1 + 6, p_4 = p_1 + 8$.

Consider the remainders of $p_1, p_1 + 2, p_1 + 6, p_1 + 8$ when divided by 5. If $p_1 \equiv 2 \pmod{5}$, then $p_1 + 8$ is divisible by 5 and hence is not a prime. If $p_1 \equiv 3 \pmod{5}$, then $p_1 + 2$ is divisible by 5. If $p_1 \equiv 4 \pmod{5}$, then $p_1 + 6$ is divisible by 5. Hence the only possibility is $p_1 \equiv 1 \pmod{5}$.

Thus we see that $p_1 \equiv 1 \pmod{2}$, $p_1 \equiv 2 \pmod{3}$ and $p_1 \equiv 1 \pmod{5}$. We conclude that $p_1 \equiv 11 \pmod{30}$.

Similarly $q_1 \equiv 11 \pmod{30}$. It follows that 30 divides $p_1 - q_1$.

3. Define a sequence $\langle f_0(x), f_1(x), f_2(x), \dots \rangle$ of functions by

$$f_0(x) = 1, \quad f_1(x) = x, \quad (f_n(x))^2 - 1 = f_{n+1}(x)f_{n-1}(x), \quad \text{for } n \geq 1.$$

Prove that each $f_n(x)$ is a polynomial with integer coefficients.

Solution: Observe that

$$f_n^2(x) - f_{n-1}(x)f_{n+1}(x) = 1 = f_{n-1}^2(x) - f_{n-2}(x)f_n(x).$$

This gives

$$f_n(x)(f_n(x) + f_{n-2}(x)) = f_{n-1}(x)(f_{n-1}(x) + f_{n+1}(x)).$$

We write this as

$$\frac{f_{n-1}(x) + f_{n+1}(x)}{f_n(x)} = \frac{f_{n-2}(x) + f_n(x)}{f_{n-1}(x)}.$$

Using induction, we get

$$\frac{f_{n-1}(x) + f_{n+1}(x)}{f_n(x)} = \frac{f_0(x) + f_2(x)}{f_1(x)}.$$

Observe that

$$f_2(x) = \frac{f_1^2(x) - 1}{f_0(x)} = x^2 - 1.$$

Hence

$$\frac{f_{n-1}(x) + f_{n+1}(x)}{f_n(x)} = \frac{1 + (x^2 - 1)}{x} = x.$$

Thus we obtain

$$f_{n+1}(x) = xf_n(x) - f_{n-1}(x).$$

Since $f_0(x)$, $f_1(x)$ and $f_2(x)$ are polynomials with integer coefficients, induction again shows that $f_n(x)$ is a polynomial with integer coefficients.

Note: We can get $f_n(x)$ explicitly:

$$f_n(x) = x^n - \binom{n-1}{1}x^{n-2} + \binom{n-2}{2}x^{n-4} - \binom{n-3}{3}x^{n-6} + \dots$$

4. Let ABC be a triangle. An interior point P of ABC is said to be **good** if we can find exactly 27 rays emanating from P intersecting the sides of the triangle ABC such that the triangle is divided by these rays into 27 smaller triangles of equal area. Determine the number of **good** points for a given triangle ABC .

Solution: Let P be a good point. Let l, m, n be respectively the number of parts the sides BC, CA, AB are divided by the rays starting from P . Note that a ray must pass through each of the vertices the triangle ABC ; otherwise we get some quadrilaterals.

Let h_1 be the distance of P from BC . Then h_1 is the height for all the triangles with their bases on BC . Equality of areas implies that all these bases have equal length. If we denote this by x , we get $lx = a$. Similarly, taking y and z as the lengths of the bases of triangles on CA and AB respectively, we get $my = b$ and $nz = c$. Let h_2 and h_3 be the distances of P from CA and AB respectively. Then

$$h_1x = h_2y = h_3z = \frac{2\Delta}{27},$$

where Δ denotes the area of the triangle ABC . These lead to

$$h_1 = \frac{2\Delta l}{27a}, \quad h_2 = \frac{2\Delta m}{27b}, \quad h_3 = \frac{2\Delta n}{27c}.$$

But

$$\frac{2\Delta}{a} = h_a, \quad \frac{2\Delta}{b} = h_b, \quad \frac{2\Delta}{c} = h_c.$$

Thus we get

$$\frac{h_1}{h_a} = \frac{l}{27}, \quad \frac{h_2}{h_b} = \frac{m}{27}, \quad \frac{h_3}{h_c} = \frac{n}{27}.$$

However, we also have

$$\frac{h_1}{h_a} = \frac{[PBC]}{\Delta}, \quad \frac{h_2}{h_b} = \frac{[PCA]}{\Delta}, \quad \frac{h_3}{h_c} = \frac{[PAB]}{\Delta}.$$

Adding these three relations,

$$\frac{h_1}{h_a} + \frac{h_2}{h_b} + \frac{h_3}{h_c} = 1.$$

Thus

$$\frac{l}{27} + \frac{m}{27} + \frac{n}{27} = \frac{h_1}{h_a} + \frac{h_2}{h_b} + \frac{h_3}{h_c} = 1.$$

We conclude that $l + m + n = 27$. Thus every **good** point P determines a partition (l, m, n) of 27 such that there are l, m, n equal segments respectively on BC, CA, AB .

Conversely, take any partition (l, m, n) of 27. Divide BC, CA, AB respectively in to l, m, n equal parts. Define

$$h_1 = \frac{2l\Delta}{27a}, \quad h_2 = \frac{2m\Delta}{27b}.$$

Draw a line parallel to BC at a distance h_1 from BC ; draw another line parallel to CA at a distance h_2 from CA . Both lines are drawn such that they intersect at a point P inside the triangle ABC . Then

$$[PBC] = \frac{1}{2}ah_1 = \frac{l\Delta}{27}, \quad [PCA] = \frac{m\Delta}{27}.$$

Hence

$$[PAB] = \frac{n\Delta}{27}.$$

This shows that the distance of P from AB is

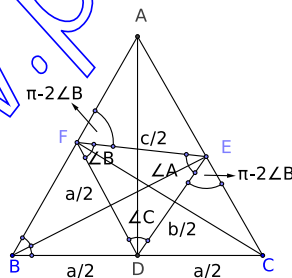
$$h_3 = \frac{2n\Delta}{27c}.$$

Therefore each triangle with base on CA has area $\frac{\Delta}{27}$. We conclude that all the triangles which partitions ABC have equal areas. Hence P is a **good** point.

Thus the number of **good** points is equal to the number of positive integral solutions of the equation $l + m + n = 27$. This is equal to

$$\binom{26}{2} = 325.$$

5. Let ABC be an acute-angled triangle, and let D, E, F be points on BC, CA, AB respectively such that AD is the median, BE is the internal angle bisector and CF is the altitude. Suppose $\angle FDE = \angle C$, $\angle DEF = \angle A$ and $\angle EFD = \angle B$. Prove that ABC is equilateral.



Solution: Since $\triangle BFC$ is right-angled at F , we have $FD = BD = CD = a/2$. Hence $\angle BFD = \angle B$. Since $\angle EFD = \angle B$, we have $\angle AFE = \pi - 2\angle B$. Since $\angle DEF = \angle A$, we also get $\angle CED = \pi - 2\angle B$. Applying sine rule in $\triangle DEF$, we have

$$\frac{DF}{\sin A} = \frac{FE}{\sin C} = \frac{DE}{\sin B}.$$

Thus we get $FE = c/2$ and $DE = b/2$. Sine rule in $\triangle CED$ gives

$$\frac{DE}{\sin C} = \frac{CD}{\sin(\pi - 2B)}.$$

Thus $(b/\sin C) = (a/2 \sin B \cos B)$. Solving for $\cos B$, we have

$$\cos B = \frac{a \sin c}{2b \sin B} = \frac{ac}{2b^2}.$$

Similarly, sine rule in $\triangle AEF$ gives

$$\frac{EF}{\sin A} = \frac{AE}{\sin(\pi - 2B)}.$$

This gives (since $AE = bc/(a + c)$), as earlier,

$$\cos B = \frac{a}{a + c}.$$

Comparing the two values of $\cos B$, we get $2b^2 = c(a + c)$. We also have

$$c^2 + a^2 - b^2 = 2ca \cos B = \frac{2a^2c}{a + c}.$$

Thus

$$4a^2c = (a + c)(2c^2 + 2a^2 - 2b^2) = (a + c)(2c^2 + 2a^2 - c(a + c)).$$

This reduces to $2a^3 - 3a^2c + c^3 = 0$. Thus $(a - c)^2(2a + c) = 0$. We conclude that $a = c$. Finally

$$2b^2 = c(a + c) = 2c^2.$$

We thus get $b = c$ and hence $a = c = b$. This shows that $\triangle ABC$ is equilateral.

6. Let $f : \mathbb{Z} \rightarrow \mathbb{Z}$ be a function satisfying $f(0) \neq 0$, $f(1) = 0$ and

(i) $f(xy) + f(x)f(y) = f(x) + f(y)$;

(ii) $(f(x - y) - f(0))f(x)f(y) = 0$,

for all $x, y \in \mathbb{Z}$, simultaneously.

(a) Find the set of all possible values of the function f .

(b) If $f(10) \neq 0$ and $f(2) = 0$, find the set of all integers n such that $f(n) \neq 0$.

Solution: Setting $y = 0$ in the condition (ii), we get

$$(f(x) - f(0))f(x) = 0,$$

for all x (since $f(0) \neq 0$). Thus either $f(x) = 0$ or $f(x) = f(0)$, for all $x \in \mathbb{Z}$. Now taking $x = y = 0$ in (i), we see that $f(0) + f(0)^2 = 2f(0)$. This shows

that $f(0) = 0$ or $f(0) = 1$. Since $f(0) \neq 0$, we must have $f(0) = 1$. We conclude that

either $f(x) = 0$ or $f(x) = 1$ for each $x \in \mathbb{Z}$.

This shows that the set of all possible value of $f(x)$ is $\{0, 1\}$. This completes (a).

Let $S = \{n \in \mathbb{Z} \mid f(n) \neq 0\}$. Hence we must have $S = \{n \in \mathbb{Z} \mid f(n) = 1\}$ by (a). Since $f(1) = 0$, 1 is not in S . And $f(0) = 1$ implies that $0 \in S$. Take any $x \in \mathbb{Z}$ and $y \in S$. Using (ii), we get

$$f(xy) + f(x) = f(x) + 1.$$

This shows that $xy \in S$. If $x \in \mathbb{Z}$ and $y \in \mathbb{Z}$ are such that $xy \in S$, then (ii) gives

$$1 + f(x)f(y) = f(x) + f(y).$$

Thus $(f(x) - 1)(f(y) - 1) = 0$. It follows that $f(x) = 1$ or $f(y) = 1$; i.e., either $x \in S$ or $y \in S$. We also observe from (ii) that $x \in S$ and $y \in S$ implies that $f(x - y) = 1$ so that $x - y \in S$. Thus S has the properties:

- (A) $x \in \mathbb{Z}$ and $y \in S$ implies $xy \in S$;
- (B) $x, y \in \mathbb{Z}$ and $xy \in S$ implies $x \in S$ or $y \in S$;
- (C) $x, y \in S$ implies $x - y \in S$.

Now we know that $f(10) \neq 0$ and $f(2) = 0$. Hence $f(10) = 1$ and $10 \in S$; and $2 \notin S$. Writing $10 = 2 \times 5$ and using (B), we conclude that $5 \in S$ and $f(5) = 1$. Hence $f(5k) = 1$ for all $k \in \mathbb{Z}$ by (A).

Suppose $f(5k + l) = 1$ for some l , $1 \leq l \leq 4$. Then $5k + l \in S$. Choose $u \in \mathbb{Z}$ such that $lu \equiv 1 \pmod{5}$. We have $(5k + l)u \in S$ by (A). Moreover, $lu = 1 + 5m$ for some $m \in \mathbb{Z}$ and

$$(5k + l)u = 5ku + lu = 5ku + 5m + 1 = 5(ku + m) + 1.$$

This shows that $5(ku + m) + 1 \in S$. However, we know that $5(ku + m) \in S$. By (C), $1 \in S$ which is a contradiction. We conclude that $5k + l \notin S$ for any l , $1 \leq l \leq 4$. Thus

$$S = \{5k \mid k \in \mathbb{Z}\}.$$

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