Pre Olympiad Solution 2016

1. **Solution:**

Let the collection of the numbers be $a_1, a_2, a_3, \ldots\ldots a_{2016}$

Now according to question

Sum of any 1008 integers is positive

$\therefore\ a_1 + a_2 + a_3 \ldots\ldots + a_{1008}$ is positive ....(1)

Also $a_{1009} + a_{1010} \ldots\ldots + a2016$ is positive ....(2)

Adding (1) & (2)

(As we know sum of 2 positive quantities is positive.

$\therefore\ a_1 + a_2 \ldots\ldots a_{1008} + a_{1009} \ldots\ldots a_{2016}$ is positive.

Hence proved.

2. **Solution:**

Let the guests be

$G_1, G_2, G_3, G_4, G_5, G_6, G_7, G_8, G_9, G_{10}$ & their respective shoe sizes be

$S_1 < S_2 < S_3 < S_4 < S_5 < S_6 < S_7 < S_8 < S_9 < S_{10}$

[Assuming all guest have different shoe sizes]

Now If G_i $(i \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\})$

spends the night there it means that all $S_j(j \geq i)$ home been taken.

Let G_i be the guest with lowest shoes size who stays the night store.

No of people who have to stag is & No of people who have gone are $10 - i$

No of shoes taken

$i \leq 10 - i$

$2i \leq 5$

$i \leq 5$

$\therefore$ Maximum value of is 5.

3. **Solution:**

$ab + cd/a$

$\Rightarrow$ a = x(ab + cd) for integer x

Similarly b = y (ad + cb) ..(2)

c = z (ad + bc) ..(3)

d = b(ad + bc) ...(4)

Multiply (1) & (4)

$ad = xv\ (ad + bc)^2$ ...(5)

Multiply (2) & (3)

$bc = yz\ (ad + bc)^2$ ...(6)

Adding (5) & (6)

$ad + bc = (xv + yz)\ (ad + bc)^2$

1 = (xv + yz) (ab + bc)

As (xv + yz) & (ad + bc)are integers & then product is 1.

$\therefore\ (xv + yz) = (ad + bc) = \pm 1$

Hence proved.

4. $(4 \cos^2 9^0 - 3)\ (4\cos^2 27^0 - 3) = \tan 9^0$

LHS = $(4(\cos^2 9^0) - 3)\ (4\ (\cos^2 27^0) - 3)$

= (2 + 2cos 18 – 3) (2 + 2cos 54 – 3)

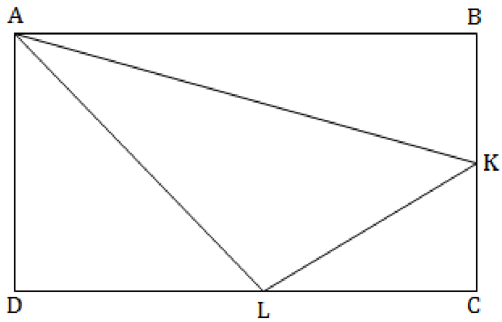
= (2 cos 18 – 1) (2 cos 54 – 1)

= (0.902104) (0.1649034)

= 0.14876

RHS = tan 9 = $\frac{\sin 9}{\cos 9} = \frac{2\sin 9}{2\cos 9} = 15838$

5.



Let AB = CD = a & AD = BC = b

Also let BK = $\frac{b}{x}$ for some real number x

Now Ar $\Delta ABK = $ Ar ΔADL

$\frac{1}{2}a\frac{b}{x} = \frac{1}{2}b\,DL$

= $\boxed{DL = \frac{b}{x}}$

KC = BC – BK & $LC = \frac{a(x-1)}{x}$

= $\frac{b(x-1)}{x}$

$\therefore$ Ar $\Delta KCL = \frac{1}{2} \times KC \times LC$

= $\frac{ab}{2}\left[\left(\frac{x-1}{x}\right)\right]^2$

Ar $\Delta AKL =$ Ar ABCD – Ar ABK – ArADL – ArKCL

= $ab - \frac{ab}{2x} - \frac{ab}{2x} - \frac{ab}{2x}\frac{(x-1)^2}{x}$

= $\frac{ab}{2x^2}\left(2x^2 - x - x - x^2 - 1 + 2x\right)$

$\boxed{Ar\,\Delta AKL = \frac{ab}{2x^2}\left(x^2-1\right)}$

Ar $\Delta AKL = Ar\Delta ABK$

= $\frac{ab}{2x^2}\left(x^2-1\right) = \frac{ab}{2x}$

= $x^2 - 1 = x$

= $x^2 - x - 1 = D$

$\Rightarrow\ x = \dfrac{1 \pm \sqrt{+2}}{2} = \dfrac{1 \pm \sqrt{5}}{2}$

Ar $\dfrac{1-\sqrt{5}}{2}$ is negative & length cannot be negative

$\therefore\ x = \dfrac{1+\sqrt{5}}{2}$

$\therefore\ \dfrac{b}{x} = b \times \dfrac{2}{1+\sqrt{5}} = \dfrac{2b\left(\sqrt{5}-1\right)}{5-1}$

$= b\dfrac{\left(\sqrt{5}-1\right)}{2}$

$\therefore$ K is $b\dfrac{\left(\sqrt{5}-1\right)}{2}$ units from B on BC

L is a $\left(\dfrac{\sqrt{5}-1}{2}\right)$units from D on CD

6. Given $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}<1$

So, a, b, c $\neq$ 1

So, to make the inequality true, we must find the maximum value of $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}$

So Now, taking a, b, c difference

$\Rightarrow$ a, b, c = 2, 3, 4

$\dfrac{1}{2}+\dfrac{1}{3}+\dfrac{1}{4} = \dfrac{6+4+3}{12} = \dfrac{13}{12} \leq 1$

$\Rightarrow$ a, b, c = 2, 3, 5

$\dfrac{1}{2}+\dfrac{1}{3}+\dfrac{1}{5} = \dfrac{6+15+10}{30} = \dfrac{31}{30} \leq 1$

$\Rightarrow$ a, b, c = 2, 3, 6

$\dfrac{1}{2}+\dfrac{1}{3}+\dfrac{1}{6} = \dfrac{3+6+9}{18} = \dfrac{1}{1} \leq 1$

$\Rightarrow$ a, b, c = 2, 3, 7

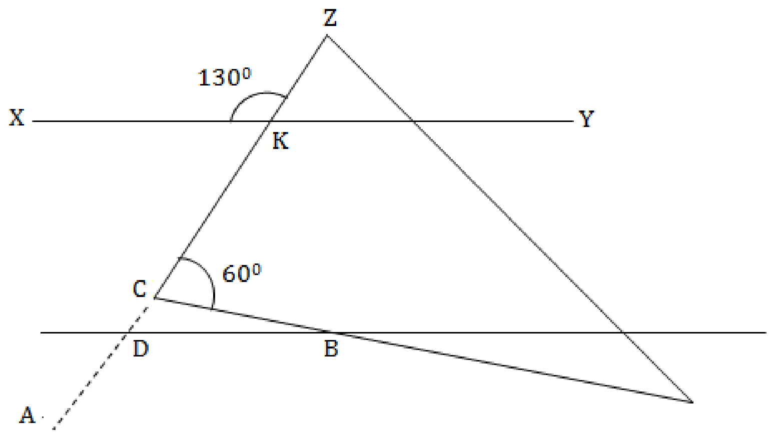
$= \dfrac{1}{2}+\dfrac{1}{3}+\dfrac{1}{7} = \dfrac{21+14+6}{42} = \dfrac{41}{42} < 1$

So, the maximum value of $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}$

they obtained is $\dfrac{41}{42}$

Now so, $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c} \leq \dfrac{41}{42}$

7.



Construction: Produced ZC

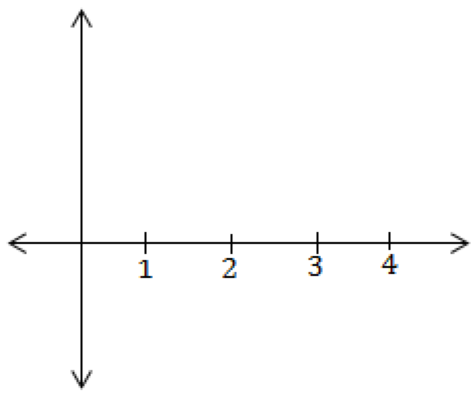
Intersection AB at D

$\angle$ CDA = 180^0

$\angle$ BCD = 120^0

$\therefore$ By extension C properly $\angle$ ABC = 10^0.

8.



$f(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f.$

$f(\alpha) = 1$

$f(\beta) = 2$

$f(\gamma) = 3$

$f(\eta) = 5$ $\quad (\eta \text{ is unique})$