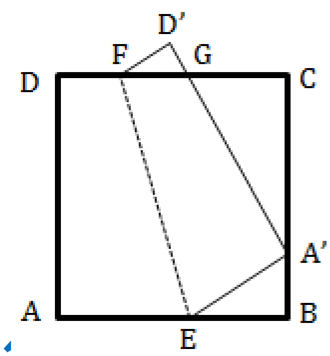
32^{nd} Indian National Mathematical Olympiad-2017

Time : 4 hours **January 15, 2017**

Instructions:

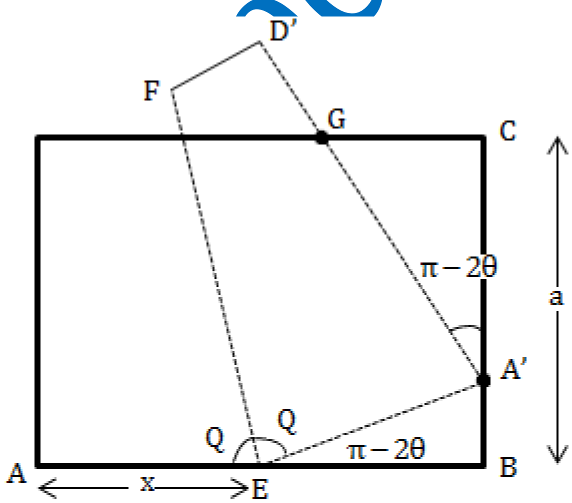
- Calculators (in any form) and protractors are not allowed.
- Rulers and compasses are allowed.
- All questions carry equal marks. Maximum marks: 102.
- Answer all the questions.
- Answer to each question should start on a new page. Clearly indicate the question number.

1. In the given figure, ABCD is a square sheet of paper. It is folded along EF such that A goes to a point A' different from B and C, on the side BC and D goes to D'. The line A'D' cuts CD in G. Show that the inradius of the triangle GCA' is the sum of the inradii of the triangles GD'F and A'BE.



Solution:

(Using S.O.T.)



$\because$ $\Delta GCA'$, $\Delta GD'F$ & $\Delta A'BE$ are similar, then

It is sufficient to prove that

$$R_{\Delta GLA'} = R_{GD'F} + R_{\Delta A'BE}$$

$\Rightarrow \quad \boxed{A'G = FG + A'E}$

Let AE $= x$ % $\angle FEA = Q$, $\quad \Rightarrow \quad A'E = x$ ...(1)

in $\Delta A'EB$, $BE = A'E \cos(\pi - 2\theta)$

$\Rightarrow$ side of a square $= x - x\cos 2\theta$

$\Rightarrow \boxed{a = 2x \sin\theta}$

Also $A'G = -\dfrac{A'C}{\cos^2\theta} = \dfrac{x\sin 2\theta - a}{\cos^2\theta}$

$\Rightarrow \quad A'G = \dfrac{2x\sin\theta}{\cos\theta + \sin\theta}$ ...(2)

Similarly $D'G = a - A'G = \dfrac{x\sin 2\theta(\sin\theta - \cos\theta)}{(\sin\theta + \cos\theta)}$

$\Rightarrow FG = \dfrac{D'G}{\sin 2\theta} = \dfrac{x(\sin\theta - \cos\theta)}{(\cos\theta + \sin\theta)}$ ...(3)

Apparently FG + A'E = $\dfrac{2x\sin\theta}{\cos\theta + \sin\theta}$

$\Rightarrow \boxed{FG + A'E = A'G}$

2. Suppose $n \ge 0$ is an integer and all the roots of $x^3 + ax + 4 - (2 \times 2016^n) = 0$ are integers. Find all possible values of α.

Solution:

Let roots be a, b, c

So, $abc = 2[2016^x - 2]$

Also $a + b + c = 0$

So, $a^3 + b^3 + c^3 = 3abc$

$\Rightarrow$ $a^3 = b^3 + c^3 = 6[2016^x - 2]$

Now for $n \ge 1$, R.H.S is

$-3 \bmod 9$ & $a^3 \equiv 0$ or $\pm 1 \bmod 9$

So, $a^3 \equiv b^3 \equiv c^3 = -1 \bmod 9$

i.e., a, b, c $\in \{9k+2, 9k+5, 9k+8\}$

Also $a + b + c \equiv 0 \bmod 9$

which is not possible so, (n = 0)

& $x^3 + \alpha x + z = 0$ has 2 inequal roots, 1, 1 & – 2, so

$\boxed{\alpha = 1 - 2 - 2 = -3}$

3. Find the number of triples (x, a, b) where x is a real number and a, b belong to the set {1, 2, 3, 4, 5, 6, 7, 8, 9} such that $x^2 - a(x) + b = 0$, where {x} denotes the fractional part of the real number x.

(For example {1.1} = 0.1 = {– 0.9}.)

Solution:

$x^2 - a\{x\} + b = 0 \Rightarrow \quad \{x\} = \frac{x^2 + b}{a}$

$\Rightarrow 0 \le \frac{x^2 + b}{a} < 1 \quad \Rightarrow \quad 0 \le x^2 < a - b \le 8$

So $\boxed{x \in [-2\sqrt{2}, 2\sqrt{3}]}$

Now, Case-1: If $x \in [0,1)$

$\Rightarrow x^2 - ax + b = 0$; f(o). f(1)<0

($\because$ Both roots can't be less than 1)

$\Rightarrow \boxed{a - b \ge 1}$ (28 solutions)

Case -2: If $x \in [1,2]$

$\Rightarrow x^2 - ax + (a + b) = 0$; f(1). f(2) < 0

$\Rightarrow$ (1 + h) (4 – a + b) < 0 $\Rightarrow \boxed{a - b > 4}$ (10 solve)

(Again Both roots can't lie in [1, 2), check)

Case 3: If $x \in [2,3)$

$\Rightarrow x^2 - ax + (2a + b) = 0$; f(2). f(3) < 0

$\Rightarrow \boxed{(a - b) > 9}$ (Not possible)

Case-4: If $x \in [-1,0)$

$\Rightarrow x^2 - ax - (a - b) = 0$; f(1). f(0) < 0

$\Rightarrow \boxed{(a - b) > 0}$ (36 solutions)

Case -5: If $x \in [-2, -1]$

$\Rightarrow x^2 - ax - 2a + b = 0$: f(–2). f(–1) < 0

$\Rightarrow$ (4 + b) (1 – a +b) < 0 $\Rightarrow \boxed{a - b > 1}$ (28 solutions)

Case-6: If $x \in [-3,-2)$; f(–3) f(–2) < 0

$\Rightarrow x^2 - ax - 3a + b = 0 \Rightarrow \boxed{(a-b) > 4}$ (10 solutions)

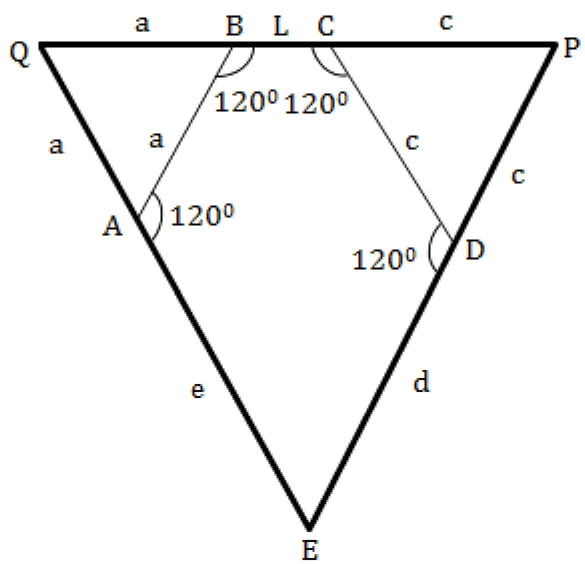
Total solutions : 28 + 10 + 36 + 10 + 28

= $\boxed{1/2 \text{ solutions}}$

4. Let ABCDE be a convex pentagon in which $\angle A = \angle B = \angle C = \angle D = 120^0$ and side lengths are five consecutive inters in some order. Find all possible values of AB + BC + CD.

Solution:

Using Pure Geometry



Construction shows ΔCPD, ΔBQA & ΔPEQ are equilateral triangles.

Now Let AB = a, BC = c, CD = c, DE = d, EA = e

$\Rightarrow a + b + c = a + e = c + d$

$\Rightarrow d = a + n \;\&\; e = b + c$

So sides are a, b, c, a + b, b + c

W.L.D.G Let + a < c

Case – 1 If c > a + b, order of sides must be b, a , a + b, c, b + c; obviously b = 1, a = 2, c = 4

Case-2 If c < a + b, order of sides will be b, a, c, a + b, b + c or a, c, b, a + b, b + c

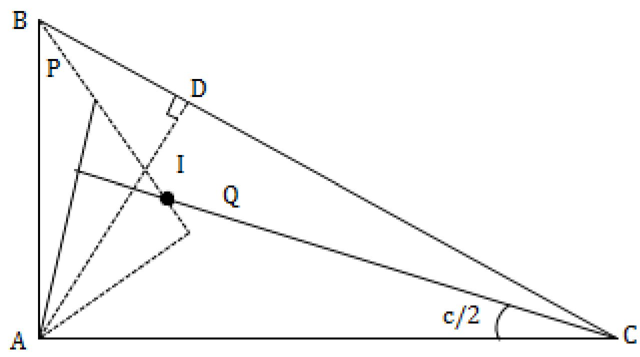
So b = 2 or 3. If b = 2, a = 3, c = 4 & If b = 3, a = 1, c = 2

So (a + b + c) can be 6, 7 or 9.

5. Let ABC be a triangle with $\angle A = 90^0$ and AB < AC. Let AD be the altitude from A on to BC. Let P, Q and I denote respectively the incentres of triangles ABD, ACD and ABC. Prove that AI is perpendicular to PQ and AI = PQ.

Solution:

Using Pure Geometry



At the Diagram shows, AP, AQ, CQ are angle bisectors of $\angle BAD$, $\angle DAC$ & $\angle BCA$ respectively.

So, $\angle PAC = \dfrac{90^0 - \angle B}{2} + 90^0 - \angle C$

$\Rightarrow \angle PAC + \angle QCA = \dfrac{90^0 - \angle B}{2} + \dfrac{90^0 - \angle B}{2} + 90^0 - \angle C + \dfrac{\angle C}{2} = 90^0$

$\Rightarrow CQ \perp AP$, Similarly $BP \perp AQ$

So I is orthocentre of ΔPAQ & $\boxed{AI \perp PQ}$ ....(i)

Now I in orthocentre of ΔPAQ & $\angle A = 45^0$

So $\dfrac{PQ}{2R'} = \sin\dfrac{\pi}{4}$ (Sine Rule) & R' = Circumradius of ΔPAQ)

$\Rightarrow PQ = R'\sqrt{2}$

Also distance of orthocenter from vertex A

$IA = 2R'\cos\dfrac{\pi}{4} \Rightarrow R'\sqrt{2}$

$\Rightarrow \boxed{PQ = AI}$

6. Let $n \geq 1$ be an integer and consider the sum

$$x = \sum_{k \geq 0} \binom{n}{2k} 2^{n-2k} 3^k = \binom{n}{0} 2^{n-2}.3 + \binom{n}{4} 2^{n-4}.3^2 + \ldots.$$

Show that 2x – 1, 2x, 2x + 1 form the sides of a triangle whose area and inradius are also integers.

Solution:

$$x = \frac{(2+\sqrt{3})^n + (2-\sqrt{3})^n}{2} \quad (= \text{Integer} > 1) \because$$

(i) Triangular inequality -

(2x – 1) + 2x > 2x + 1 $\Rightarrow \boxed{x > 1}$

(ii) Area = $\sqrt{s(s-a)(s-b)(s-c)}$ (there S = 3x)

= $\sqrt{3x(x+1).\, x\,(x-1)}$

= $x\sqrt{3(x^2-1)}$

$$\left\{x^2 - 1 = \left(\frac{(2+\sqrt{3})^n - (2-\sqrt{3})^n}{2}\right)\right\}$$

$$\Rightarrow \text{Area} = \left[\frac{(2+\sqrt{3})^n + (2-\sqrt{3})^n}{2}\right]\left[\frac{(2+\sqrt{3})^n - (2-\sqrt{3})^n}{2}\right]\sqrt{3}$$

(Binomial exp. Shows (it is an integer)

(iii) Inradius = $\frac{\Delta}{S} = \frac{(2+\sqrt{3})^n - (2-\sqrt{3})^n}{2\sqrt{3}} = \text{Integer}$ (Using Binomial Expansion)