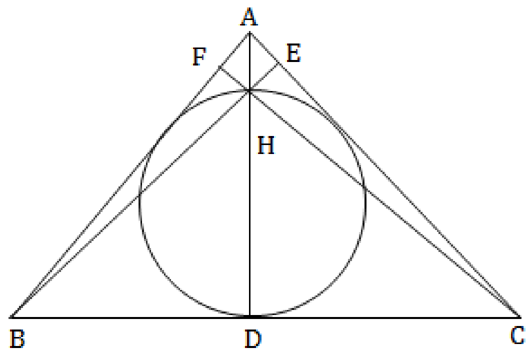
INMO-2016

1. Let ABC be triangle in which AB = AC. Suppose the orthocentre of the triangle lies on the in-circle. Find the ratio AB/BC.

Solution:

Since the triangle is isosceles, the orthocentre lies on the perpendicular AD from A on to BC. Let it cut the in-circle at H. Now we are given that H is the orthocentre of the triangle. Let AB = AC = b and BC = 2a. Then BD = a. Observe that b > a since b is the hypotenuse and a is a leg of a right-angled triangle. Let BH meet AC in E and CH meet AB in F. By Pythagoras theorem applied to ΔBDH, we get



$BH^2 = HD^2 + BD^2 = 4r^2 + a^2$,

where r is the in-radius of ABC. We want to compute BH in another way. Since A, F, H, E are con-cyclic, we have

$BH.\ BE = BF\ .\ BA.$

But $BF.\ BA = BD.\ BC = 2a^2$, since A, F, D, C are con-cyclic. Hence $BH^2 = 4a^4/BE^2$. But

$$BE^2 = 4a^2 - CE^2 = 4a^2 - BF^2 = 4a^2 - \left(\frac{2a^2}{b}\right)^2 = \frac{4a^2\left(b^2 - a^2\right)}{b^2}.$$

This leads to

$$BH^2 = \frac{a^2b^2}{b^2 - a^2}.$$

Thus we get

$$\frac{a^2b^2}{b^2 - a^2} = a^2 + 4r^2.$$

This simplifies to $(a^4/(b^2 - a^2)) = 4r^2$. Now we relate a, b, r in another way using area. We know that $[ABC] = rs$, where s is the semi-perimeter of ABC We have $s = (b + b + 2a)/2 = b + a$. On the other hand area can be calculated using Heron's formulae:

$$[ABC]^2 = s(s - 2a)\,(s - b)\,(s - b) = (b + a)\,(b - a)a^2 = a^2(b^2 - a^2).$$

Hence

$$r^2 = \frac{[ABC]^2}{s^2} = \frac{a^2(b^2 - a^2)}{(b + a)^2}.$$

Using this we get

$$\frac{a^4}{b^2 - a^2} = 4\left(\frac{a^2\left(b^2 - a^2\right)}{(b + a)^2}\right).$$

Therefore $a^2 = 4(b - a)^2$, which gives $a = 2(b - a)$ or $2b = 3a$. Finally,

$$\frac{AB}{BC}=\frac{b}{2a}=\frac{3}{4}.$$

$$\frac{a^4}{b^2-a^2}=4\left(\frac{a^2(b^2-a^2)}{(b+a)^2}\right).$$

Therefore $a^2 = 4(b - a)^2$, which gives $a = 2(b - a)$ or $2b = 3a$. Finally, $\frac{AB}{BC}=\frac{b}{2a}=\frac{3}{4}$.

Alternate Solution 1:

We use the known facts BH = 2R cos B and r = 4R sin (A/2) sin (B/2) sin (C/2), where R is the circumradius of ΔABC and r its in-radius. Therefore

$$HD = BH \sin \angle HBD = 2R \cos B \sin\left(\frac{\pi}{2}-C\right)=2R\cos^2 B,$$

since $\angle C = \angle B$. But $\angle B = (\pi-\angle A)/2$, since ABC is isosceles. Thus we obtain

$$HD = 2R\cos^2\left(\frac{\pi}{2}-\frac{A}{2}\right).$$

However HD is also the diameter of the in circle. Therefore HD = 2r. Thus we get

$$2R\cos^2\left(\frac{\pi}{2}-\frac{A}{2}\right)=2r=8R\sin(A/2)\sin^2((\pi-A)/4).$$

This reduces to

sin (A/2) = 2(1 – sin (A/2)).

Therefore sin (A/2) = 2/3. We also observe that sin (A/2) = BD/AB. Finally

$$\frac{AB}{BC}=\frac{AB}{2BD}=\frac{1}{2\sin(A/2)}=\frac{3}{4}.$$

Alternate Solution 2:

Let D be the mid-point of BC. Extend AD to meet the circumcircle in L. Then we know that HD = DL. But HD = 2r. Thus DL = 2r. Therefore IL = ID + DL = r +2r = 3r. We also know that LB = LI. Therefore LB = 3r. This gives

$$\frac{BL}{LD}=\frac{3r}{2r}=\frac{3}{2}.$$

But ΔBLD is similar to ΔABD. So

$$\frac{AB}{BD}=\frac{BL}{LD}=\frac{3}{2}.$$

Finally,

$$\frac{AB}{BC}=\frac{AB}{2BD}=\frac{3}{4}.$$

Alternate solution 3:

Let D be the mid-point of BC and E be the mid-point of DC. Since DI = IH (= r) and DE =EC, the mid-point theorem implies that IE ∥ CH. But CH $\perp$ AB. Therefore EI $\perp$ AB. Let EI meet AB in F. Then F is the point of tangency of the incircle of ΔABC with AB. Since the incircle is also tangent to BC at D, we have BF = BD. Observe that ΔBFE is similar to ΔBDA. Hence

$$\frac{AB}{BD}=\frac{BE}{BF}=\frac{BE}{BD}=\frac{BD+DE}{BD}=1+\frac{DE}{BD}=\frac{3}{2}.$$

This gives

$$\frac{AB}{BC} = \frac{3}{4}.$$

2. For positive real numbers a, b, c which of the following statements necessarily implies a = b = c: (I) $a(b^3 + c^3) = b(c^3 + a^3) = c(a^3 + b^3)$, (II) $a(a^3 + b^3) = b(b^3 + c^3) = c(c^3 + a^3)$? Justify your answer.

Solution:

We show that (I) need not imply that a = b = c where as (II) always implies a = b = c.

Observe that $a(b^3 + c^3) = b(c^3 + a^3)$ gives $c^3(a - b) = ab(a^2 - b^2)$. This gives either a = b or $ab(a + b) = c^3$. Similarly, b = c or $bc(b + c) = a^3$. If $a \neq b$ and $b \neq c$, we obtain

$$ab(a + b) = c^3, \quad bc(b + c) = a^3.$$

Therefore

$$b(a^2 - c^2) + b^2(a - c) = c^3 - a^3.$$

This gives $(a - c)(a^2 + b^2 + c^2 + ab + bc + ca) = 0$. Since a, b, c are positive, the only possibility is a = c. We have therefore 4 possibilities: a = b = c; $a \neq b$, $b \neq c$ and c = a; $b \neq c$, $c \neq a$ and a = b; $c \neq a$, $a \neq b$ and b = c.

Suppose a = b and b, $a \neq c$. Then $b(c^3 + a^3) = c(a^3 + b^3)$ gives $ac^3 + a^4 = 2ca^3$. This implies that $a(a - c)(a^2 - ac - c^2) = 0$. Therefore $a^2 - ac - c^2 = 0$. Putting a/c = x, we get the quadratic equation $x^2 - x - 1 = 0$. Hence $x = (1 + \sqrt{5})/2$. Thus we get

$$a = b = \left(\frac{1+\sqrt{5}}{2}\right)c, \quad \text{c arbitrary positive real number.}$$

Similarly, we get other two cases:

$$b = c = \left(\frac{1+\sqrt{5}}{2}\right)a, \quad \text{a arbitrary positive real number;}$$

$$c = a = \left(\frac{1+\sqrt{5}}{2}\right)b, \quad \text{b arbitrary positive real number.}$$

And a = b = c is the fourth possibility.

Consider (II) : $a(a^3 + b^3) = b(b^3 + c^3) = c(c^3 + a^3)$. Suppose a, b, c are mutually distinct. We may assume a = max {a, b, c}. Hence a > b and a > c. Using a > b, we get from the first relation that $a^3 + b^3 < b^3 + c^3$. Therefore $a^3 < c^3$ forcing a < c. This contradicts a > c. We conclude that a, b, c cannot be mutually distinct. This means some two must be equal. If a = b, the equality of the first two expressions give $a^3 + b^3 = b^3 + c^3$ so that a = c. Similarly, we can show that b = c implies b = a and c = a gives c = b.

Alternate for (II) by a contestant: We can write

$$\frac{a^3}{c} + \frac{b^3}{c} = \frac{c^3}{a} + a^2,$$

$$\frac{b^3}{a} + \frac{c^3}{a} = \frac{a^3}{b} + b^2,$$

$$\frac{c^3}{b} + \frac{a^3}{b} = \frac{b^3}{c} + c^2.$$

Adding, we get

$$\frac{a^3}{c} + \frac{b^3}{a} + \frac{c^3}{b} = a^2 + b^2 + c^2.$$

Using C-S inequality, we have

$$\left(a^2+b^2+c^2\right)^2=\left(\frac{\sqrt{a^3}}{\sqrt{c}}.\sqrt{ac}+\frac{\sqrt{b^3}}{\sqrt{a}}.\sqrt{ba}+\frac{\sqrt{c^3}}{\sqrt{b}}.\sqrt{cb}\right)^2\le\left(\frac{a^3}{c}+\frac{b^3}{a}+\frac{c^3}{b}\right)(ac+ba+cb)$$

$= (a^2 + b^2 + c^2)(ab + bc + ca)$.

Thus we obtain

$$a^2 + b^2 + c^2 \le ab + bc + ca.$$

However this implies $(a - b)^2 + (b - c)^2 + (c - a)^2 \le 0$ and hence $a = b = c$.

3. Let $\mathbb{N}$ denote the set of all natural numbers. Define a function $T : \mathbb{N} \to \mathbb{N}$ by $T(2k) = k$ and $T(2k + 1) = 2k + 2$. We write $T^2(n) = T(T(n))$ and in general $T^k(n) = T^{k-1}(T(n))$ for any $k > 1$.

(i) Show that for each $n \in \mathbb{N}$ there exists k such that $T^k(n) = 1$.

(ii) For $k \in \mathbb{N}$, let c_k denote the number of elements in the set $\{n : T^k(n) = 1\}$. Prove that $c_{k+2} = c_{k+1} + c_k$, for $k \ge 1$.

Solution:

(i) For $n = 1$, we have $T(1) = 2$ and $T^2(1) = T(2) = 1$. Hence we may assume that $n > 1$.

Suppose $n > 1$ is even. Then $T(n) = n/2$. We observe that $(n/2) \le n - 1$ for $n > 1$.

Suppose $n > 1$ is odd so that $n \ge 3$. Then $T(n) = n + 1$ and $T^2(n) = (n + 1)/2$. Again we see that $(n + 1)/2 \le (n - 1)$ for $n \ge 3$.

Thus we see that in at most $2(n - 1)$ steps T sends n to 1. Hence $k \le 2(n - 1)$. (Here $2(n - 1)$ is only a bound. In reality, less number of steps will do.)

(ii) We show that $c_n = f_{n+1}$, where f_n is the n-th Fibonacci number.

Let $n \in \mathbb{N}$ and let $k \in N$ be such that $T^k(n) = 1$. Here n can be odd or even. If n is even, it can be either of the form $4d + 2$ or of the form $4d$.

If n is odd, then $1 = T^k(n) = T^{k-1}(n + 1)$. (Observe that $k > 1$; otherwise we get $n + 1 = 1$ which is impossible since $n \in \mathbb{N}$). Here $n + 1$ is even.

If $n = 4d + 2$, then again $1 = T^k(4d + 2) = T^{k-1}(2d + 1)$. Here $2d + 1 = n/2$ is odd.

Thus each solution of $T^{k-1}(m) = 1$ produces exactly one solution of $T^k(n) = 1$ and n is either odd or of the form $4d + 2$.

If $n = 4d$, we see that $1 = T^k(4d) = T^{k-1}(2d) = T^{k-2}(d)$. This shows that each solution of $T^{k-2}(m) = 1$ produces exactly one solution of $T^k(n) = 1$ of the form $4d$.

Thus the number of solutions of $T^k(n) = 1$ is equal to the number of solutions of $T^{k-1}(m) = 1$ and the number of solutions of $Tk^{-2}(l) = 1$ for $k > 2$. This shows that $c_k = c_{k-1} + c_{k-2}$ for $k > 2$. We also observe that 2 is the only number which goes to 1 in one step and 4 is the only number which goes to 1 in two steps. Hence $c_1 = 1$ and $c_2 = 2$. This proves that $c_n = f_{n+1}$ for all $n \in \mathbb{N}$.

4. Suppose 2016 points of the circumference of a circle are coloured red and the remaining points are coloured blue. Given any natural number $n \ge 3$, prove that there is a regular n-sides polygon all of whose vertices are blue.

Solution:

Let $A_1, A_2, ..., A_{2016}$ be 2016 points on the circle which are coloured red and the remaining blue.

Let $n \ge 3$ and let $B_1, B_2, ..., B_n$ be a regular n-sided polygon inscribed in this circle with the vertices chosen in anti-clock-wise direction. We place B_1 at A_1. (It is possible, in this position, some other B's also coincide with some other A's). Rotate the polygon in antic-clock-wise direction gradually till some B's coincide with (an equal number of) A's second time. We again rotate the polygon in the same direction till some B's coincide with an equal number of A's third time, and so on until we return to the original position, i.e., B_1 at A_1. We see that the number of rotations will not be more than $2016 \times n$, that is, at most these many times some B's would have coincided with an equal number of A's. Since the interval $(0, 360^0)$ has infinitely many points, we can find a value $\alpha^o \in (0, 360^0)$ through which the polygon can be rotated from its initial position such that no B coincides with any A.

This gives a n-sided regular polygon having only blue vertices.

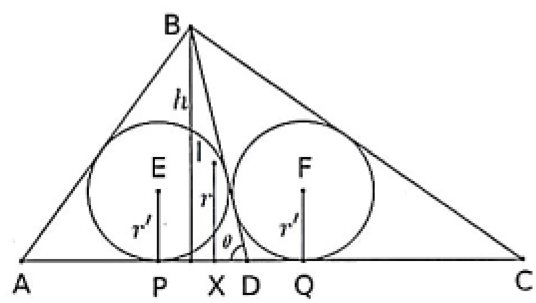
Alternate Solution: Consider a regular $2017 \times n$-gon on the circle; say, $A_1A_2A_3.....A_{2017n}$. For each $j, 1 \le j \le 2017$, consider the points $\{A_k : k \equiv j \pmod{2017}\}$. These are the vertices of a regular n-gon, say S_j. We get 2017 regular n-gons; $S_1, S_2, ..., S_{2017}$. Since there are only 2016 red points, by pigeon-hole principle there must be some n-gon among these 2017 which does not contain any red point. But then it is a blue n-gon.

5. Let ABC be a right-angled triangle with $\angle B = 90^0$. Let D be a point on AC such that the in-radii of the triangles ABD and CBD are equal. If this common value is r' and if r is the in-radius of triangle ABC, prove that

$$\frac{1}{r'} = \frac{1}{r} + \frac{1}{BD}.$$

Solution:

Let E and F be the incentres of triangles ABD and CBD respectively. Let the incircles of triangles ABD and CBD touch AC in P and Q respectively. If $\angle BDA = \theta$, we see that



$$r' = PD\tan(\theta/2) = QD\cot(\theta/2).$$

Hence

$$PQ = PD + QD = r'\left(\cot\frac{\theta}{2} + \tan\frac{\theta}{2}\right) = \frac{2r'}{\sin\theta}.$$

But we observe that

$$DP = \frac{BD + DA - AB}{2}, DQ = \frac{BD + DC - BC}{2}.$$

Thus PQ = $(b - c - a + 2BD)/2$. We also have

$$\frac{ac}{2} = [ABC] = [ABD] + [CBD] = r'\frac{(AB + BD + DA)}{2} + r'\frac{(CB + BD + DC)}{2} = r'\frac{(c + a + b + 2BD)}{2} = r'(s + BD).$$

But

$$r' = \frac{PQ\sin\theta}{2} = \frac{PQ.h}{2BD},$$

where h is the altitude from B on to AC. But we know that $h = ac/b$. Thus we get

$$ac = 2\times r'(s+BD) = 2\times\frac{PQ.h}{2\times BD}(s+BD) = \frac{(b-c-a+2BD)ca(s+BD)}{2\times BD\times b}.$$

Thus we get

$$2 \times BD \times b = 2 \times (BD - (s - b))\,(s + BD).$$

This gives $BD^2 = s(s - b)$. Since ABC is a right-angled triangle $r = s - b$. Thus we get $BD^2 = rs$.
On the other hand, we also have $[ABC] = r'(s + BD)$. Thus we get

$$rs = [ABC] = r'(s+BD).$$

Hence

$$\frac{1}{r'} = \frac{1}{r} + \frac{BD}{rs} = \frac{1}{r} + \frac{1}{BD}.$$

Alternate Solution 1: Observe that

$$\frac{r'}{r} = \frac{AP}{AX} = \frac{CQ}{CX} = \frac{AP+CQ}{AC},$$

where X is the point at which the incircle of ABC touches the side AC. If s_1 and s_2 are respectively the semi-perimeters of triangles ABD and CBD, we show $AP = s_1 - BD$ and $CQ = s_2 - BD$.
Therefore

$$\frac{r'}{r} = \frac{(s_1-BD)+(s_2-BD)}{AC} = \frac{s_1+s_2-2BD}{b}.$$

But

$$s_1+s_2 = \frac{AD+BD+c}{2} + \frac{CD+BD+a}{2} = \frac{(a+b+c)+2BD}{2} = \frac{s+BD}{2}.$$

This gives

$$\frac{r'}{r} = \frac{s+BD-2BD}{b} = \frac{s-BD}{b}.$$

We also have

$$r' = \frac{[ABD]}{s_1} = \frac{[CBD]}{s_2} = \frac{[ABD]+[CBD]}{s_1+s_2} = \frac{[ABC]}{s+BD} = \frac{rs}{s+BD}.$$

This implies that

$$\frac{r'}{r} = \frac{s}{s+BD}.$$

Comparing the two expressions for r'/r, we see that

$$\frac{s-BD}{b} = \frac{s}{s+BD}.$$

Therefore $s^2 - BD^2 = bs$, or $BD^2 = s\,(s - b)$. Thus we get $BD = \sqrt{s(s-b)}$.

We know now that

$$\frac{r'}{r} = \frac{s}{s+BD} = \frac{s-BD}{b} = \frac{BD}{(s-b)+BD} = \frac{\sqrt{s(s-b)}}{(s-b)+\sqrt{s(s-b)}} = \frac{\sqrt{s}}{\sqrt{s-b}+\sqrt{s}}.$$

Therefore $\dfrac{r}{r'} = 1+\sqrt{\dfrac{s-b}{s}}.$

This gives $\dfrac{1}{r'} = \dfrac{1}{r} + \left(\sqrt{\dfrac{s-b}{s}}\right)\dfrac{1}{r}.$

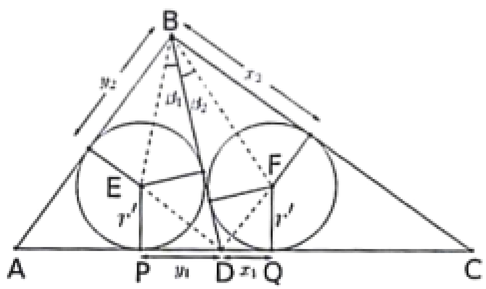
But

$$\left(\sqrt{\frac{s-b}{s}}\right)\frac{1}{r} = \left(\frac{s-b}{\sqrt{s(s-b)}}\right)\frac{1}{r} = \left(\frac{s-b}{BD}\right) = \frac{1}{r}.$$

If $\angle B = 90^0$, we know that r = s – b. Therefore we get

$$\frac{1}{r'} = \frac{1}{r} + \left(\frac{s-b}{BD}\right)\frac{1}{r} = \frac{1}{r} + \frac{1}{BD}.$$

Alternate solution 2 by a contestant: Observe that $\angle EDF = 90^0$. Hence ΔEDP is similar to ΔDFQ. Therefore DP. DQ = EP. FQ. Taking DP = y_1 and DQ = x_1, we get $x_1 y_1 = (r')^2$. We also observe that BD = $x_1 + x_2 = y_1 + y_2$. Since $\angle EBF = 45^0$, we get



$$1 = \tan 45^0 = \tan(\beta_1 + \beta_2) = \frac{\tan\beta_1 + \tan\beta_2}{1 - \tan\beta_1 \tan\beta_2}.$$

But $\tan\beta_1 = r'/y_2$ and $\tan\beta_2 = r'/x_2$. Hence we obtain

$$1 = \frac{(r'/y_2) + (r'/x_2)}{1 - (r')^2 / x_2 y_2}.$$

Solving for r' , we get

$$r' = \frac{x_2 y_2 - x_1 y_1}{x_2 + y_2}.$$

We also know

$$r = \frac{AB + BC - AC}{2} = \frac{x_2 + y_2 - (x_1 + y_1)}{2} = \frac{(x_2 - x_1) + (y_2 - y_1)}{2}.$$

Finally,

$$\frac{1}{r} + \frac{1}{BD} = \frac{2}{(x_2 - x_1) + (y_2 - y_1)} + \frac{1}{x_1 + x_2}$$

$$= \frac{2x_1 + 2x_2 + (x_2 - x_1) + (y_2 - y_1)}{(x_1 + x_2)((x_2 + x_1) + (y_2 + y_1))}.$$

But we can write

$2x_1 + 2x_2 + (x_2 - x_1) + (y_2 - y_1) = (x_1 + x_2 + x_2 - x_1) + (y_1 + y_2 + y_2 - y_1) = 2(x_2 + y_2),$

and

$(x_1 + x_2)((x_2 - x_1) + (y_2 - y_1)) = 2(x_1 + x_2)(x_2 - y_1)$
$= 2(x_2(x_2 + x_1 - y_1) - x_1 y_1) = 2(x_2 y_2 - x_1 y_1).$

Therefore

$$\frac{1}{r}+\frac{1}{BD}=\frac{2(x_2+y_2)}{2(x_2y_2-x_1y_1)}=\frac{1}{r'}.$$

Remark: One can also choose B = (0, 0), A = (0, a) and C = (1, 0) and the coordinate geometry proof gets reduced considerably.

6. Consider a non-constant arithmetic progression $a_1, a_2, \ldots, a_n, \ldots$. Suppose there exist relatively prime positive integers p > 1 and q > 1 such that a_1^2, a_{p+1}^2 and a_{q+1}^2 are also the terms of the same arithmetic progression. Prove that the terms of the arithmetic progression are all integers.

Solution:

Let us take $a_1 = a$. We have

$$a^2 = a + kd, \quad (a + pd)^2 = a + ld, \quad (a + qd)^2 = a + md.$$

Thus we have

$$a+ld=(a+pd)^2=a^2+2pad+p^2d^2=a+kd+2pad+p^2d^2.$$

Since we have non-constant AP, we see that $d\neq 0$. Hence we obtain $2pa + p^2d = l - k$. Similarly, we get $2qa + q^2d = m - k$. Observe that $p^2q - pq^2 \neq 0$. Otherwise p = q and gcd (p, q) = p > 1which is a contradiction to the given hypothesis that gcd(p, q) = 1. Hence we can solve the two equations for a, d:

$$a=\frac{p^2(m-k)-q^2(l-k)}{2(p^2q-pq^2)},\ d=\frac{q(l-k)-p(m-k)}{p^2q-pq^2}.$$

It follows that a, d are rational numbers. We also have

$$p^2a^2 = p^2a + kp^2d.$$

But $p^2d = l - k - 2pa$. Thus we get

$$p^2a^2 = p^2a + k(l - k - 2pa) = (p - 2k)\, pa + k(l - k).$$

This shows that pa satisfies the equation

$$x^2 - (p - 2k)x - k(l - k) = 0.$$

Since a is rational, we see that pa is rational. Write pa = w/z, where w is an integer and z is a natural numbers such that gcd (w, z) = 1. Substituting in the equation, we obtain

$w^2 - (p - 2k)wz - k(l - k)z^2 = 0.$

This shows z divides w. Since gcd(w, z) = 1, it follows that z = 1 and pa = w an integer. (In fact any rational solution of a monic polynomial with integer coefficients is necessarily an integer). Similarly, we can prove that qa is an integer. Since gcd (p, q) = 1, there are integers u and v such that pu + qv = 1. Therefore a = (pa)u + (qa) v. It follows that a is an integer.

But $p^2d = l - k - 2pa$. Hence p^2d is an integer. Similarly, q^2d is also an integer. Since gcd $(p^2, q^2) = 1$, it follows that d is an integer. Combining these two, we see that all the terms of the AP are integers.

Alternatively, we can prove that a and b are integers in another way. We have seen that a and d are rationals; and we have three relations:

$a^2 = a + kd, \quad p^2d + 2pa = n_1, \quad q^2d + 2qa = n_2,$

where $n_1 = l - k$ and $n_2 = m - k$. Let a = u/v and d = x/y where u, x are integers and v, y are natural numbers, and gcd (u, y) = 1, gcd (x, y) = 1. Putting this in these relations, we obtain

$$u^2y = uvy + kxv^2, \quad (1)$$
$$2puy + p^2vx = vyn_1, \quad (2)$$
$$2quy + q^2vx = vyn_2. \quad (3)$$

Now (1) shows that v/u^2y. Since gcd (u, v) = 1, it follows that v/y. Similarly (2) shows that y/p^2vx. Using gcd (y, x) = 1, we get that y/p^2v. Similarly, (3) shows that y/q^2v. Therefore y divides gcd $(p^2v, q^2v) = v$. The two results $v|y$ and $y|v$ imply v = y, since both v, y are positive.

Substitute this in (1) to get

$$y^2 = uv + kxv.$$

This shows that $v|u^2$. Since gcd (u, v) = 1, it follows that v= 1. This gives v = y = 1. Finally a = u and d = x which are integers.