

**IMO 2016**

1. Triangle BCF has a right angle at B. Let A be the point on line CF such that  $FA = FB$  and F lies between A and C. Point D is chosen such that  $DA = DC$  and AC is the bisector of  $\angle DAB$ . Point E is chosen such that  $EA = ED$  and AD is the bisector of  $\angle EAC$ . Let M be the mid-point of CF. Let X be the point such that AMXE is a parallelogram (where  $AM \parallel EX$  and  $AE \parallel MX$ ). Prove that lines BD, FX, and ME are concurrent.
2. Find all positive integers  $n$  for which each cell of an  $n \times n$  table can be filled with one of the letters I, M and O in such a way that:
  - in each row and each column, one third of the entries are I, one third are M and one third are O; and
  - in any diagonal, if the number of entries on the diagonals is a multiple of three, then one third of the entries are I, one third are M and one third are O.**Note:** The rows and columns of an  $n \times n$  table are each labelled 1 to  $n$  in a natural order. Thus each cell corresponds to a pair of positive integers  $(i, j)$  with  $1 \leq i, j \leq n$ . For  $n > 1$ , the table has  $4n - 2$  diagonals of two types. A diagonal of the first type consists of all cells  $(i, j)$  for which  $i + j$  is a constant, and a diagonal of the second type consists of all cells  $(i, j)$  for which  $i - j$  is a constant.
3. Let  $P = A_1A_2\dots A_k$  be a convex polygon in the plane. The vertices  $A_1, A_2, \dots, A_k$  have integral coordinates and lie on a circle. Let  $S$  be the area of  $P$ . An odd positive integer  $n$  is given such that the squares of the side lengths of  $P$  are integers divisible by  $n$ . Prove that  $2S$  is an integer divisible by  $n$ .
4. A set of positive integers is called fragrant if it contains at least two elements and each of its elements has a prime factor in common with at least one of the other elements. Let  $P(n) = n^2 + n + 1$ . What is the least possible value of the positive integer  $b$  such that there exists a non-negative integer  $a$  for which the set  $\{(a + 1), P(a + 2), \dots, P(a + b)\}$  is fragrant?
5. The equation  $(x - 1)(x - 2)\dots(x - 2016) = (x - 1)(x - 2)\dots(x - 2016)$  is written on the board, with 2016 linear factors on each side. What is the least possible value of  $k$  for which it is possible to erase exactly  $k$  of these 4032 linear factors so that at least one factor remains on each side the resulting equation has no real solutions?
6. There are  $n \geq 2$  line segments in the plane such that every two segments cross, and no three segments meet at a point. Geoff has to choose an endpoint of each segment and place a frog on it, facing the other endpoint. Then he will clap his hands  $n - 1$  times. Every time he claps, each frog will immediately jump forward to the next intersection point on its segment. Frogs never change the direction of their jumps. Geoff wishes to place the frogs in such a way that no two of them will ever occupy the same intersection point at the same time.
  - (a) Prove that Geoff can always fulfill his wish if  $n$  is odd.
  - (b) Prove that Geoff can never fulfill his wish if  $n$  is even.