SET III

SECTION – A

1. Prove that $(53)^{103} + (103)^{53}$ is divisible by 39.

[4 marks]

Sol :

53 is congruent to 14 and 103 is congruent to – 14 with respect to 39, therefore replacing 53 and 103 by 14 and – 14 respectively, we get

$(14)^{103} + (-14)^{53}$

$= (14)^{103} + (-14)^{53} = 14^{53}(14^{50} - 1)$

Since 14^{53} and 39 are co-prime, therefore 14^{53} cannot be multiple of 39. So let us consider 14^{50}-1 only

$14^{50} - 1 = (14^2)^{25} - 1 = (196)^{25} - 1$

196 is congruent to 1 with respect to 39, therefore replacing 1996 by 1, we get

$(1)^{25} - 1 = 0$

(remainder when 1450^{-1} is divided by 39)

Thus $14^{50} - 1$ is divisible by 39

Hence $(53)^{103} + (103)^{53}$ is divisible by 39

2. Six persons go to a birthday party. They leave their top-coats in the lounge and pick them while returning back. In how many ways can they pick up the top-coats, so that

(i) exactly one person picks up his own top-coat;

(ii) exactly two persons pick up their own top coats

Sol :

(i) Since any one of the six persons can be the one to pick up this own top-coat, and the remaining five persons all pick-up the wrong top-coats, therefore the desired number of ways = $6\ D_5 = 6 \times 44 = 264$.

(ii) Two persons (who pick-up their own top-coats) can be selected out of 6 persons

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in ${}^6C_2$ ways. For each such selection the remaining four persons pick-up wrong top-coats in D_4 ways. Therefore by the fundamental multiplication principle, the desired number of ways

$= {}^6C_2 \times D_4 = 15 \times 9 = 135$ [4 marks]

3. A person who left home between 4 p.m. and 5 p.m. returned between 5 p.m. and 6 p.m. and found that the hands of his watch exactly changed places. When did he go out?

Sol :

The dial of a clock is divided into 60 equal divisions. In one hour the minute hand makes one complete revolution, i.e., it moves through 60 divisions, and the hour-hand moves through 5 divisions. Suppose that when the man the man went out the hour-hand was x divisions ahead of the zeroth mark (the point labelled 12 on the dial), where $20 < x < 25$. Also suppose that when the man came back, the hour-hand was y divisions ahead of the zeroth mark, when $25 < y < 30$.

Since the minute-hand and hour-hand exactly interchanged places during the interval that the man remained out, it follows that when the man went out, the minute-hand was at y and hour-hand was at x, and when the man came back the minute-hand was at x and the hour-hand was at y.

Since the minute-hand moves 12 times as fast as the hour-hand, therefore considering the positions of the hands at the time of going out and coming back separately, we have

$y = 12(x - 20)$, ...(1)

and $x = 12(y - 25)$. .....(2)

Solving the above equations for x and y, we have

$x = \frac{3180}{143}$,

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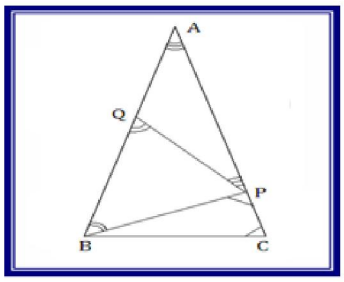
$y = \frac{3840}{143}$,

$= 26\frac{122}{143}$.

Therefore the man went out at $26\frac{122}{143}$ minutes past 4

[4 marks]

4. In Δ ABC, AB = AC, P and Q are points on AC and AB respectively such that CB = BP = PQ = QA. Then prove that $\angle AQP = \frac{5\pi}{7}$.



Sol :

[Hint : If $\angle AQP = x$, then $\angle QAP = \angle QPA = 90^0 - \frac{1}{2}x$

$\angle PQB = \angle PBQ = 180^0 - x$

$\angle BCA = \angle ABC = \angle BPC$

Since = $45^0 + \frac{1}{4}x$,

$\angle APQ + \angle QPB + \angle BPC = 180^0, - 180^0$. Therefore

$\left(90^0 - \frac{1}{2}x\right) + \left(2x - 180^0\right) + 45^0 + \frac{1}{4}x = 180^0$, so that $x = 5\pi/7$.]

[4 marks]

5. In a right triangle ABC right-angled at A, the radius of the inscribed circle is 2 cm. Further, the radius of the ex-circle touching the side BC and also the sides AB and AC produced is 15 cm. Determine the sides a, b, c of the triangle.

Sol :

Radius of the inscribed circle = $\frac{\Delta}{s} = 2$.

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Radius of ex-circle opposite A = $\frac{\Delta}{s-a}$ = 15.

$\therefore \quad \frac{s-a}{s} = \frac{2}{15}$, so that s = $\frac{15}{13}$ a.

Substituting the value of s in $\frac{\Delta}{s} = 2$, we get $\Delta = \frac{30}{13}$ a,

or $\qquad \frac{1}{2}bc = \frac{30}{13}a,$

so that $bc = \frac{60}{13}a.$

Also b + c =2s – a = 2 . $\frac{15}{13}a - a = \frac{17}{13}a.$

The equations b + c = $\frac{17}{13}a$, bc = $\frac{60}{13}$ a give

$$b = \frac{5}{13}a,\ c = \frac{12}{13}a\ ;$$

or $\qquad b = \frac{12}{13}a,\ c = \frac{5}{13}a.$

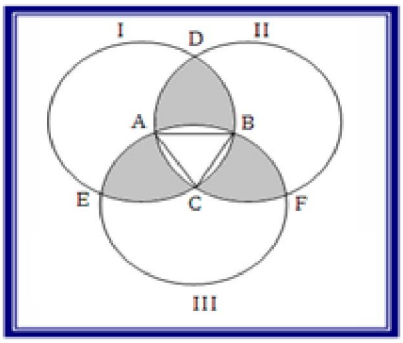
From these values of b, c we get

$\frac{\Delta}{s} = \frac{(bc)/2}{(b+c+a)/2} = \frac{2}{13}$ a, so that a = 13.

Thus the sides are 5, 12, 13

[4 marks]

6. The centre of each of three unit circles is an intersection of the remaining two as shown in the figure. Find the area of the shaded portion



Sol :

A, B, C are the centres of the three circles.

Therefore AB = BC = CA = 1.

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Also $\quad AB = AD = DB = 1.$

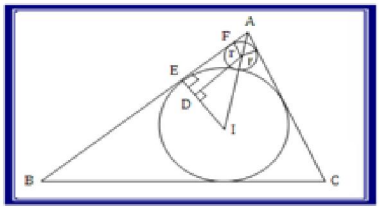
$\therefore \quad \angle BAD = 60^0.$

Also, area of the smaller segment of circle III cut off by AB = area of the smaller segment of circle II cut off by AD. Therefore we find that area of the shaded potion ABD = area of sector BAD. By symmetry, the areas of the three shaded portions AEC, BFC, ADB are equal. Therefore area of the shades portion = 3 × area of sector BAD. Since $\angle BAD = 60^0$, therefore area of the shaded portion

$= \frac{1}{2} \times$ area of one of the circles$= \frac{\pi}{2}$.

[4 marks]

7. Let ABC be a triangle and a circle r' be drawn lying inside the triangle touching the two sides AB and AC. Show that the radii of the circles r' and r is equal to $\tan^2\left(\frac{\pi - A}{4}\right)$.



Sol :

Let I be the incentre, r the inradius and E the point of contact of the incircle with AB. Also, let I' be the centre of the touching AB, AC and the incircle, r' the radius of this circle and F its point of contact with AB. Since AB and AC both touch this circle, its centre must also lie on AI. From

I' draw $I'D \perp IE$. in

$\Delta\, II'D$,

$ID = r - r'$,

$II' = r + r'$,

$\angle IDI' = \frac{\pi}{2}$,

$\angle DI'I = \frac{A}{2}$

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$$\therefore \quad \frac{r-r'}{r+r'} = \sin\left(\frac{A}{2}\right) = \cos\theta, \text{ where } \theta = \frac{\pi - A}{2}.$$

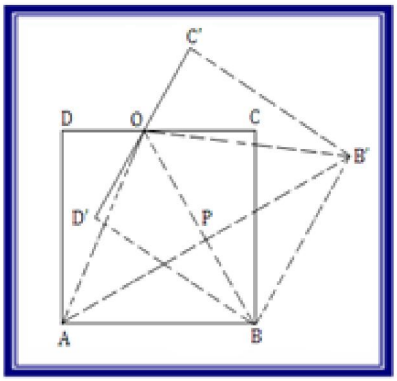
$$\therefore \quad \frac{r'}{r} = \frac{1-\cos\theta}{1+\cos\theta} = \tan^2\left(\frac{\theta}{2}\right) = \tan^2\frac{\pi - A}{4}.$$

[4 marks]

8. A rigid square plate ABCD of unit side rotates in its own plane about the middle point of CD unit the new position of A coincides with the old position of B. How far is the new position of B from old position of A?

Sol :

Let O be the mid-point of CD. Since the new position of A coincides with the old position of B, therefore, the rotation is in the counter clockwise sense about O, through the angle AOB.



Let B', C', D' be the new positions of B, C, D respectively.

Let OB and A B' intersect at P. Then $\angle BOB' = \angle AOB$.

Also OA = OB = OB'.

Therefore OB is the internal bisector of $\angle AOB'$ of isosceles triangle AOB'. Therefore $OP \perp AB'$ and AP = PB'.

Also, $[AOB] = \frac{1}{2} \times$ area of square ABCD $= \frac{1}{2}$.

But, $[AOB] = \frac{1}{2} AP.OB, OB = \sqrt{\left(OC^2 + CB^2\right)} = \frac{\sqrt{5}}{2}$.

$\therefore \quad AP = \frac{2}{\sqrt{5}}$, so that $AB' = 2AP = \frac{4}{\sqrt{5}} = \frac{4\sqrt{5}}{5}$.

[4 marks]

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9. Five men A, B, C, D, and E are wearing caps of black or white colour without each knowing the colour of his cap. It is known that a man wearing a black cap always speaks the truth while the ones wearing white tell lies. If they make the following statements, find the colour worn by each of them:

A: I see three black caps and one white.

B: I see four white caps.

C: I see one black cap and three white.

D: I see four black caps.

Sol :

Suppose A is speaking the truth, Then he must he wearing a black cap, so that four persons are wearing black caps and one person is wearing a white cap. The statements made by B and C are both in contradiction with this statement. Therefore B and C must be wearing white caps. This implies that A' s statement is false. The contradiction show that A is telling a lie, and therefore A is wearing a white cap.

Suppose now that B is speaking the truth. Then B must be wearing a black cap, and the caps of C, D, E must be all white. Since B's cap is black and caps of A, D, E are white, therefore C's statement is true and consequently his cap must be black. We have a contradiction, which shows that B is telling a lie, and therefore B is wearing a white cap.

Since A and B are both wearing white caps, therefore D is telling a lie. Therefore we find that D is wearing a white cap.

If C is speaking the truth, then E's cap must be black and C's own cap must also be black . If on the other hand, C is telling a lie, then E's cap must be white (because that is the only way in which his statement can be false) and C's cap must also be white. This implies all the five caps are white. But this means that B is speaking the truth and so his cap must be black. The contradiction shows that C's statement cannot be false. Thus C is speaking the truth, and the caps of C and E must be both black.

Hence A, B, D are wearing white caps, and C, E are wearing black caps.

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Aliter. Since E has not made any statement, we shall start with the colour of E's cap.

Two different cases arise : E's cap is either white or black.

Let us first suppose that E is a wearing a white cap.

Since D cannot see four black caps (at least E's cap is white), therefore D must be telling a lie and colour of his cap must be white.

Since D and E have white caps, A's statement is false and his cap is also white.

Suppose now that B's cap is black. Then he is speaking the truth and C's cap is white. But the caps of A, D, E being white and that of B being black (he is speaking the truth), C's statement is true and his cap must be black. This is a contradiction and consequently B's cap must be white.

If B's cap is white , he is telling a lie and so C's cap must be black (since caps of A, D, E are already white). Since C's cap is black, B's cap must be black because C sees one black cap. We again have a contradiction and consequently B's cap cannot be white.

From the a contradiction and consequently B's cap cannot be white. From the above we conclude that E's cap cannot be white.

Now E's cap being black, B cannot see four white caps, and consequently his own cap must be white.

Since B's cap is white, D cannot see four black caps and therefore his own cap must be white.

Since caps of B and D are white, A cannot see three black caps. Consequently his own cap must be white.

Now caps of A, B, D are white and that of E is black, therefore C's statement is true and his own cap must be black.

Thus A, B and D are wearing white caps, and C and E are wearing black caps

[4 marks]

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10. Find the least natural number whose last digit is 7 such that it becomes 5 times larger when this last digit is carried to the beginning of this number.

Sol :

Suppose the given number is $a_n a_{n-1} a_{n-2} \ldots a_1 7$,

i.e., $a_n.10^n + a_{n-1}.10^{n-1} + \ldots + 10a_1 + 7$.

i.e., 10x + 7, where x = $a_n 10^{n-1} + a_{n-1} 10^{n-2} + \ldots + a_1$.

When the last digit is carried to the beginning of the number, it becomes

$7\, a_n a_{n-1} \ldots a_1$,

i.e., $7.10^n + a_n 10^{n-1} + a_{n-2} 10^{n-2} + \ldots + a_1$,

i.e., $7.10^n + x$.

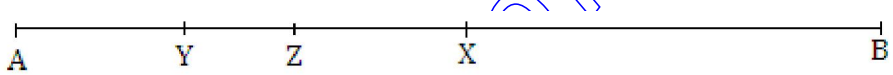
We are given that $7.10^n + x = 5(10x + 7)$,

so that $7x = 10^n - 5$. ....(i)

We shall find the least value of n, and consequently that of x as well, for which (i) has an integer solution for x.

The smallest value of n for which 10^n leaves a remainder 5 when divided by 7 turns out to be 5 (by actually dividing 1000....0 by 7. In fact 100000 = (14285) × 7+ 5.

The smallest number in than 142857.



[4 marks]

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SECTION – B

11. Prove that for any natural number n, the expression

$A = 2903^n - 803^n - 464^n + 261^n$ is divisible by 1897.

Sol:

Let us first observe that 1897 = 271 × 7. Since 271 and 7 are prime to each other (actually they are both primes), therefore it is enough to show that A is divisible by 7 as well as by 271. Let us write $A = (2903^n - 803^n) - (464^n - 261^n)$.

Since $a^n - b^n$ is always divisible by a – b, therefore $2903^n - 803^n$ is divisible by 2903 – 803 = 2100 and $464^n - 261^n$ is divisible by 464 – 261, i.e., 203. Since 2100 and 203 are both divisible by 7, it follows that A is divisible by 7.

Again, let us write $A = (2903^n - 464^n)(803^n - 261^n)$.

Now $2903^n - 464^n$ is divisible by 2903 – 464, i.e., 2439, which is a multiple of 271, so that $2903^n - 464^n$ is divisible by 271.

Also $803^n - 261^n$ is divisible by 803 – 261, i.e., 542 which is multiple of 271, so that $803^n - 261^n$ is a multiple of 271.

Since $2903^n - 464^n$ and $803^n - 261^n$ are both multiples of 271, therefore A is a multiple of 271 × 7, i.e., 1897

[6 marks]

12. Solve the equation

$3x^4 - 40x^3 + 130x^2 - 120x + 27 = 0$,

given that the product of two of its roots is equal to the product of the other two.

Sol :

Let the roots of the given equation be α, β, γ, δ.

Then

$$\sigma_1 = (\alpha + \beta) + (\gamma + \delta) = \frac{40}{3}, \qquad \text{....(i)}$$

$\sigma_2 \equiv (\alpha+\beta)(\gamma+\delta)+\alpha\beta+\gamma\delta=\frac{130}{3}$, ...(ii)

$\sigma_3 \equiv \alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta)=40$, ...(iii)

$\sigma_4 \equiv \alpha\beta.\ \gamma\delta=9$. ....(iv)

Since the product of two of the roots is equal to the product of the other two, therefore,

$\alpha\beta=\gamma\delta$. ....(v)

From (iii) and (v), we have

$\alpha\beta(\alpha+\beta+\gamma+\delta)=40$. ....(vi)

From (i), (v) and (vi), we have

$\alpha\beta=\gamma\delta=3$. ....(vii)

From (ii) and (vii), we have

$(\alpha+\beta)(\gamma+\delta)=\frac{112}{3}$. ...(viii)

From (i) and (viii), we find that $\alpha+\beta$ and $\gamma+\delta$ are the roots of the equation

$t^2-\frac{40}{3}t+\frac{112}{3}=0$.

Solving the above equation, we have

t = 4, 28/3.

Therefore, $\alpha+\beta, \gamma+\delta = 4, 28/3$. ...(ix)

From(vii) and (ix), we find that two of the numbers $\alpha, \beta, \gamma, \delta$ are the roots of the equation

$y^2 - 4y + 3 = 0$. ...(x)

and the remaining two are the roots of the equation

$y^2 - (28/3)y + 3 = 0$. ...(xi)

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Solving (x) and (xi), we find that the roots of the given equation are $1, 3, \frac{1}{3}, 9$.

[6 marks]

13. If $\alpha+\beta+\gamma=1,$

$\alpha^2+\beta^2+\gamma^2=2,$

$\alpha^3+\beta^3+\gamma^3=3,$

find the value of $\alpha^4+\beta^4+\gamma^4$.

Sol:

We shall first determine the equation whose roots are α,β,γ and then find the sum of the fourth powers of the roots of the same.

Let the equation whose roots are α,β,γ be

$x^3 + px^2 + qx + r = 0.$ ..(i)

Then $\alpha+\beta+\gamma=-p,$ ...(ii)

$\alpha\beta+\beta\gamma+\gamma\alpha=q,$ ...(iii)

$\alpha\beta\gamma=-r.$ ...(iv)

Substituting the value of $\alpha+\beta+\gamma$ in (ii), we have

$p = -1$ ...(v)

Also , by substituting the values of $\alpha+\beta+\gamma$ and $\alpha^2+\beta^2+\gamma^2$ in the identity

$$(\alpha+\beta+\gamma)^2=\sum\alpha^2+2\sum\alpha\beta,$$

we have $q=\sum\alpha\beta=-\frac{1}{2}$ ...(vi)

Since α,β,γ are the roots of (i), by substituting α,β,γ for x in (i) in succession and adding, we have

$S_3 + pS_2 + qS_1 + 3r = 0$

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Substituting $S_1 = 1$, $S_2 = 2$, $S_3 = 3$, $p = -1$ and $q = -\frac{1}{2}$ in ...(vii)

we have

$$r = -\frac{1}{6}. \qquad \text{...(viii)}$$

From (i), (v), (vi) and (viii), we find that α, β, γ are the roots of the equation

$$x^3 - x^2 - \frac{1}{2}x - \frac{1}{6} = 0. \qquad \text{...(ix)}$$

Multiplying (ix) throughout by x, we have

$$x^4 - x^3 - \frac{1}{2}x^2 - \frac{1}{6}x = 0. \qquad \text{...(x)}$$

Since α, β, γ satisfy (x), therefore, by substituting α, β, γ for x in (x) and adding, we get

$$S_4 - S_3 - \left(\frac{1}{2}\right)S_2 - \left(\frac{1}{6}\right)S_1 = 0,$$

or $$S_4 = S_3 + \frac{1}{2}S_2 + \frac{1}{6}S_1 = 3 + \frac{1}{2}2 + \frac{1}{6}1 = \frac{25}{6}.$$

[6 marks]

14. Determine all functions f satisfying the functional relation

$$f(x) + f\left(\frac{1}{1-x}\right) = \frac{2(1-2x)}{x(1-x)},$$

where x is a real number, $x \neq 0, x \neq 1$, [Here $f : R - \{0, 1\} \to R$.]

Sol:

We are given that

$$f(x) + f\left(\frac{1}{1-x}\right) = \frac{2(1-2x)}{x(1-x)} = \frac{2}{x} - \frac{2}{1-x}, \qquad \text{..... (1)}$$

For all real values of x other than 0 and 1.

Let us introduce another variable y = $\frac{1}{1-x}$ $\left(\text{so that } x = 1 - \frac{1}{y}\right)$.

From (1), we have

$$f(x) + f(y) = \frac{2}{x} - \frac{2}{y} \qquad \text{..... (2)}$$

(2) holds for all values of x and y except 0 and 1. (See what happens for y = 0 or 1.)

Also, we have

$$f(y) + f\left(\frac{1}{1-y}\right) = \frac{2}{y} - \frac{2}{1-y} \qquad \text{..... (3)}$$

for all real values of y other than 0 and 1. (This is simply (1)).

Introduce another variable z = $\frac{1}{1-y}$ $\left(\text{so that } y = 1 - \frac{1}{z}\right)$. In the same manner as above, we have

$$f(y) + f(z) = \frac{2}{y} - 2z \qquad \text{.....(4)}$$

whenever y, z $\neq$ 0 or 1.

The relations y = $\frac{1}{1-x}$ and z = $\frac{1}{1-y}$, give x = $\frac{1}{1-z}$, so that z = 1 - $\frac{1}{x}$.

Consider the relation

$$f(z) + f\left(\frac{1}{1-z}\right) = \frac{2}{z} - \frac{2}{1-z}, \qquad \text{.....(5)}$$

Substituting x = $\frac{1}{1-z}$, we have

$$f(z) + f(x) = \frac{2}{z} - 2x \qquad \text{.....(6)}$$

whenever x, z $\neq$ 0 or 1.

Adding corresponding sides of (2), (4) and (6) and dividing throughout by 2,

we have

$$f(x) + f(y) + f(z) = \left[\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)-(x+y+z)\right].$$

By (4), we get

$$f(x)=\left(\frac{1}{x}+\frac{1}{z}-\frac{1}{y}\right)-(x+y-z)$$

$$=\left(\frac{1}{x}-x\right)-\left(y+\frac{1}{y}\right)+\left(z+\frac{1}{z}\right),$$

$$=\left(\frac{1}{x}-x\right)-\left\{\frac{1}{1-x}+1-x\right\}+\left\{1-\frac{1}{x}+\frac{x}{x-1}\right\}$$

$$=\frac{x+1}{x-1}.$$

Thus $f(x) = \frac{x+1}{x-1}$ for all real values of x except x = 0 or 1.

[6 marks]

15. (i) If the internal bisectors of the base angles of a triangle be equal, prove that the triangle is isosceles. **(Lehmus Steiner Theorem)** [3 marks]

(ii) Prove that in a cyclic quadrilateral ABCD,

AB. CD + BC. AD = AC. AD. **(Ptolemy's Theorem)**

Sol:

(i) Suppose BE and CF, the internal bisectors of ∠s B and C, are equal.

$$BE = \frac{4ca\,s(s-b)}{(c+a)^2},$$

$$CF = \frac{4\,abs(s-c)}{(a+b)^2}.$$

$$\therefore \frac{4cas(s-b)}{(c+a)^2} = \frac{4abs(s-c)}{(a+b)^2}$$

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$\Rightarrow$ c $(a + b)^2$ $(s - b)$ = b $(c + a)^2$ $(s - c)$,

$\Rightarrow$ s $\{c\ (a + b)^2 - b\ (c + a)^2\} + bc\ \{(c + a)^2 - (a + b)^2\} = 0$,

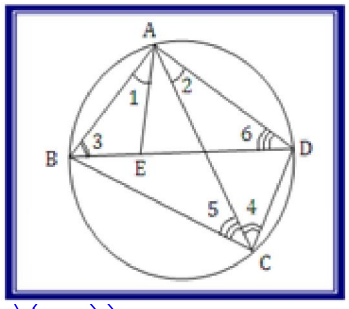
$\Rightarrow$ s $\{c(a^2 + b^2) - b(c^2 + a^2)\} + bc\ (c - b)\ (2a + b + c) = 0$,

$\Rightarrow$ s $(c - b)\ \{a^2 - bc\} + bc\ (c - b)\ (2s\ +a) = 0$.

$\Rightarrow$ $(c - b)\ \{bcs + a^2\ s + abc\} = 0$,

$\Rightarrow$ c = b, since bc s + a^2 s + abc > 0

[3 marks]



(ii) Choose a point E in BD so that $\angle$BAE = $\angle$DAC. In Δs ABE and ACD,

$\angle 1 = \angle 2$ (by const.)

$\angle 3 = \angle 4$ (angles in the same segment of a circle).

Therefore the Δs are equiangular and hence similar.

Consequently,

$$\frac{BE}{DC} = \frac{AB}{AC},$$

i.e., AB . CD = AC. BE. ... (1)

Let us now consider triangles BAC and DAE.

$\angle$BAC = $\angle$EAD (Add $\angle$EAC to equal $\angle$s 1 and 2).

$\angle 5 = \angle 6$ (angles in the same segment of a circle).

Therefore the triangles are equiangular and hence similar.

Consequently,

$$\frac{BC}{ED} = \frac{AC}{AD},$$

i.e., BC . AD = AC. ED. ... (2)

Adding corresponding sides of (1) and (2), we have

AB . CD + BC . AD = AC. BE + AC. ED = AC . BD [3 marks]

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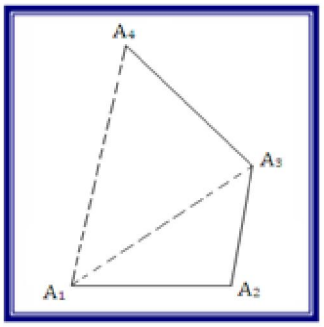
16. Suppose A_1A_2 …… A_n is an n-sided regular polygon such that

$$\frac{1}{A_1A_2}=\frac{1}{A_1A_3}+\frac{1}{A_1A_4}.$$



Determine n, the number of sides of the polygon.

Sol:

Let each side of the polygon be of length a. Since

$\angle A_1A_2A_3=(n-2)\frac{\pi}{n}$,

$A_1A_2 = A_2A_3 = a$,

therefore from $\Delta A_1A_2A_3$,

$A_1A_3 = 2a\cos(\pi/n)$.

Also, from $\Delta A_2A_3A_4$,

$A_2A_4 = 2a\cos(\pi/n)$.

Since the polygon $A_1A_2 \ldots A_n$ is regular,

its vertices lie on a circle. In particular, the quadrilateral $A_1A_2A_3A_4$ is cyclic.

By Ptolemy's theorem,

$A_1A_2 . A_3A_4 + A_1A_4 . A_2A_3 = A_1A_3 . A_2A_4$.

$\therefore\ a^2 + A_1A_4 . A_2A_3 = [2a\cos(\pi/n)]^2$,

i.e., $A_1A_4 = a(4\cos^2\pi/n - 1)$.

Since $\frac{1}{A_1A_2}=\frac{1}{A_1A_3}+\frac{1}{A_1A_4}$,

therefore,

$$\frac{1}{a}=\frac{1}{2a\cos(\pi/n)}+\frac{1}{a[4\cos^2(\pi/n)-1]},$$

$\Rightarrow 2\cos(\pi/n)[4\cos^2(\pi/n)-1] = 4\cos^2(\pi/n)-1+2\cos(\pi/n)$,

$\Rightarrow 8\cos^3(\pi/n) - 4\cos^2(\pi/n) - 4\cos(\pi/n)+1=0$,

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$\Rightarrow$ $x = \cos \pi / n$ is a root of the equation

$8x^3 - 4x^2 - 4x + 1 = 0.$ ..... (1)

We shall show that the roots of (1) are $\cos \pi/7,\ \cos 3\pi/7,\ \cos 5\pi/7.$

Let $7\theta = (2n+1)\pi$, so that $4\theta = (2n+1)\pi - 3\theta.$

$\therefore\ \cos 4\theta = \cos[(2n+1)\pi - 3\theta] = -\cos 3\theta,$

$\Rightarrow\ 2\cos^2 2\theta - 1 = -(4\cos^3\theta - 3\cos\theta),$

$\Rightarrow\ 2(2\cos^2\theta - 1)^2 - 1 + 4\cos^3\theta - 3\cos\theta = 0,$

$\Rightarrow\ 8\cos^4\theta + 4\cos^3\theta - 8\cos^2\theta - 3\cos\theta + 1 = 0,$

$\Rightarrow\ (\cos\theta + 1)[8\cos^3\theta - 4\cos^2\theta - 4\cos\theta + 1] = 0.$ .....(2)

Now (2) is satisfied by $\theta = \pi/7, 3\pi/7, 5\pi/7, \pi,$ .....

Rejecting the factor $\cos\theta + 1$ which corresponds to $\theta = \pi$, and putting $\cos\theta = y$, we find that

$\cos \pi/7, \cos 3\pi/7,\ 5\pi/7$ are the roots of

$8y^3 - 4y^2 - 4y + 1 = 0.$ .....(3)

Since equations (1) and (3) are the same, therefore the roots of (1) are

$\cos \pi/7,\ \cos 3\pi/7,\ \cos 5\pi/7.$ But $\cos \pi/n$ is a root of (1).

Therefore we must have $n = 7$

[6 marks]

17. Two pedestrians started simultaneously towards each other and met each other after 3 hours and 20 minutes. How much time will it take each of them to cover the whole distance if the first arrived at the place of departure of the second 5 hours later than the second arrived at the point of departure of the first.

Sol:

Suppose a pedestrian A starts from X towards Y with a velocity V_1 m/min, and B starts from Y towards X with a velocity V_2 m/min at the same instant.

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Let $V_1 + V_2 = V$, $XY = d$.

We are given that

$$\frac{d}{V} = 200, \qquad ....(1)$$

$$\frac{d}{V_1} - \frac{d}{V_2} = 300. \qquad ....(2)$$

Writing $\frac{V_1}{V} = x, \frac{V_2}{V} = y,$ we are have $x + y = 1$. ....(3)

Also from (1) and (2), we have

$$\frac{1}{x} - \frac{1}{y} = \frac{3}{2}, \qquad ....(4)$$

i.e., $2(y - x) = 3xy$.

From (3) and (4), we have

$$1 = (y+x)^2 = (y-x)^2 + 4.\frac{2}{3}(y-x),$$

so that $3(y-x)^2 + 8(y-x) - 3 = 0,$

i.e., $[3(y-x)-1][y-x+3] = 0$.

Since $y > x$, we get

$$y - x = \frac{1}{3}. \qquad ...(5)$$

From (3) and (5), we have $x = \frac{1}{3}, y = \frac{2}{3},$ i.e., $V_1 = \frac{V}{3}, V_2 = \frac{2V}{3}, \frac{d}{V_1} = 600, \frac{d}{V_2} = 300.$

Therefore to cover the whole distance A takes 10 hours and B takes 5 hours.

[6 marks]

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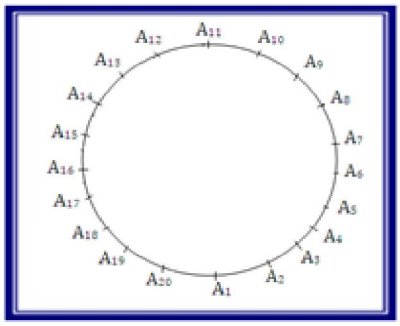
18. In a group of ten persons, each person is asked to write the sum of the ages of all the other nine persons. If all the ten sums from the nine-element set {82, 83, 84, 85, 87, 89, 90, 91, 92}, find the individual ages of the persons, assuming then to be whole numbers (of years).



Sol:

Let the ages of the ten persons (in years) be $x_1, x_2, x_3, \ldots\ldots, x_{10}$ respectively, and let $S = x_1 + x_2 + \ldots + x_{10}$. Since $x_1, x_2, \ldots\ldots, x_{10}$ are whole numbers, therefore S is also a whole number.

The sums written by the ten persons are $S - x_1, S - x_2, \ldots, S - x_{10}$, which are also whole numbers. By adding these we get the sum

$$(S - x_1) + (S - x_2) + \ldots + (S - x_{10}),$$

$$= 10S - (x_1 + x_2 + \ldots\ldots + x_{10}) = 9S,$$

since $x_1 + x_2 + \ldots + x_{10} = S$.

Therefore the sum of all ten sums written by individual persons is a multiple of 9. Since the sums $S - x_1, S - x_2, \ldots\ldots, S - x_{10}$ from a nine-elementic set, therefore two of them must be identical, say $S - x_1$ and $S - x_2$, which show that $x_1 = x_2$. The total of all the nine sums given is 82 + 83 + 84 + ... + 92 = 783, which it itself a multiple of 9. Therefore the tenth sum, which must be one of the nine given sums, must be a multiple of 9 is 90, therefore the tenth sum must be 90. Consequently the sum of all the ten sums = 783 + 90 = 873. The equation 9S = 873 gives S = 97. Subtracting the numbers 82, 83, ..., 89, 90, 91, 92 from 97, we get the individual ages to be 15, 14, 13, 12, 10, 8, 7, 7, 6, and 5 years respectively.

[6 marks]

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19. 29th February of the year 2000 will fall on a Tuesday, for your information. Show that, after this date, 29th February will fall on Tuesday s only thrice in the whole next century. What are the three years when this will happen?

Sol:

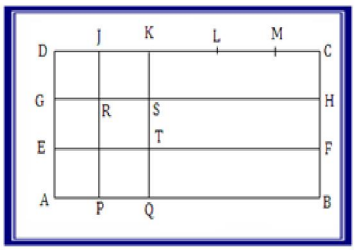
Since365 $\equiv 1$ (mod. 7), therefore 28th February (the last day of February) of 2001 will be a Wednesday (the day next to Tuesday). Let us agree to express this by saying that there is an excess of one day in any ordinary year. With this terminology, there will be an excess of two days in a leap year.

The next leap year after the year 2000 will be the year 2004. There will be 1 + 1 + 1 + 2, i.e., 5 excess days upto 29th February, 2004.

The day of the week of 29th February will be Tuesday when the number of excess day is an exact multiple of 7. Therefore our problem is to find those positive integers k for which the number of excess days in 4k years after the years 2000 is an exact multiple of 7 and for which 4k < 100. The number of excess days in 4k years is 5k. This is a multiple of 7 when k = 7, 14, 21, …., i.e., in 4 × 7, 4 × 14, 4 × 21, ….. years after the year 2000. Since only three of these numbers are less than 100, therefore in the 21st century there are only three years, namely 2028, 2056 and 2084 in which 29th February is a Tuesday

[6 marks]

20. Show how will you cut a rectangular sheet of paper along two-line segments parallel to a side and two parallel to an adjacent side, into five pieces whose areas are in the ratio 1 : 2 : 3 : 4 : 5.



Sol :

Since 1 + 2 + 3 + 4 + 5 = 15, therefore really speaking the rectangle has to be divided into 15 equal parts, and then we have to take 1, 2, 3, 4 and 5 parts respectively to get the five pieces. We take two points E and G in AD so that AE = EG = GD and draw GH, EF parallel to AB. Also, take points J, K , L, M in DC such that DJ = JK = KL = LM = MC and draw JP, KQ parallel to CD.

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Clearly areas of rectangles DJRG, GSTE ,SHFT, JCHR, EFBA are in the ratio 1 :2 : 3 : 4 : 5. The division has been made by two lines GH, EF parallel to DC, and two lines JP, KQ parallel to DA.

[6 marks]

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